Lectures on Statistical Mechanics
Grant M. Rotskoff
Last Update: March 17, 2025

# Lecture 1 Introduction

## Logistics

1. All information and announcements on the course webpage.
2. Homework is due Wendesday at 12:30pm. The solutions will be posted at class time, hence no late homework can be accepted.
3. Breakdown of grading is specified in a document on the course webpage.

## Goals of Statistical Mechanics

Statistical mechanics connects microscopic fluctuations to emergent macroscopic properties. The theory is statistical in the sense that the distribution of microscopic states provides direct information about measurable physical properties that do not appreciably fluctuate—for example, free energies, heat capacities, and equilibrium constants. The goal of this course is to introduce tools for understanding the microscopic dynamics of molecular systems and connect that dynamics to macroscopic properties. In particular, you will

1. Develop an understanding of statistical mechanical ensembles and their relation to macroscopic thermodynamic quantities
2. Connect microscopic energetics to macroscopic stability using equilibrium constants
3. Identify phase transitions and connect those transitions to fluctuations
4. Understand how molecular models are constructed and how molecular dynamics simulations are conducted
5. Differentiate between diffusion limited and activated kinetics and know how to estimate rates in both cases

## Thermodynamic States

To specify a thermodynamic state, we fix macroscopic, external variables. For example, we could specify the thermodynamic state of an ideal gas by assigning the number of particles $N$, the volume $V$, and the energy $E$. We will refer to this specification as a "macrostate". Of course, there are many different microscopic arrangements of the particles that all have the same values of $(N, V, E)$. How should we distinguish between them?

We will first say that $\nu = \{(\boldsymbol{x}_1, \boldsymbol{p}_1) \dots, (\boldsymbol{x}_N, \boldsymbol{p}_N)\}$ defines a "microstate", a complete specification of all microscopic degrees of freedom in a system. For a quantum system, would instead be a complete set of quantum numbers. The symbol $\nu$ is the Greek letter "nu".

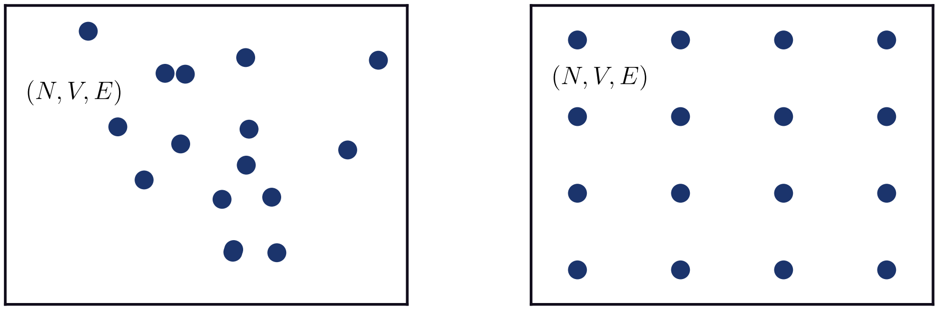


Figure 1.1: Contrasting a disordered configuration with an ordered configuration. Which feels more likely? Why?

The starting point for a statistical mechanical theory is the following assumption.

In an *isolated* system with fixed $(N, V, E)$, all *allowed* microstates are equally probable.

There are a couple of words doing a lot of the work in this definition. First, we should emphasize **isolated**; this means that the system is not directly coupled to the environment in any way—for example, there is no flow of energy through heat exchange. Obviously, such a system is difficult to prepare in laboratory settings, but to the best of our knowledge this is an accurate picture of the entire universe. Second, the word **allowed**; there are many configurations that do not satisfy the constraints, i.e., that have distinct values of $(N, V, E)$. The probability of these disallowed states is zero.

The consequences of this assumption are not entirely intuitive. For example, which of the configurations in Fig. 1.1 would you intuit is more probable? The reason that the configuration feels more likely is, of course, related to entropy. There are many more configurations that look disordered than ones that look ordered.

To see this, we can consider a simple quantum system in which particles can be either spin-up (↑) or spin-down (↓). We will assume there is no external field and that the particles are not interacting so ↑ and ↓ are equally likely. If we ask which configuration is more probable

$$\nu_1 = \downarrow\downarrow\uparrow\uparrow\downarrow\uparrow \qquad \nu_2 = \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow,$$

we immediately must deduce

$$p(\nu_1) = \left(\frac{1}{2}\right)^N = p(\nu_2), \tag{1.1}$$

which is also consistent with our initial postulate.

The possible failure of intuition arises from the fact that *the distribution of macroscopic properties is not equal to the distribution of microstates*. For example, if we introduce an observable

$$f(\nu) = \#\uparrow - \#\downarrow \tag{1.2}$$

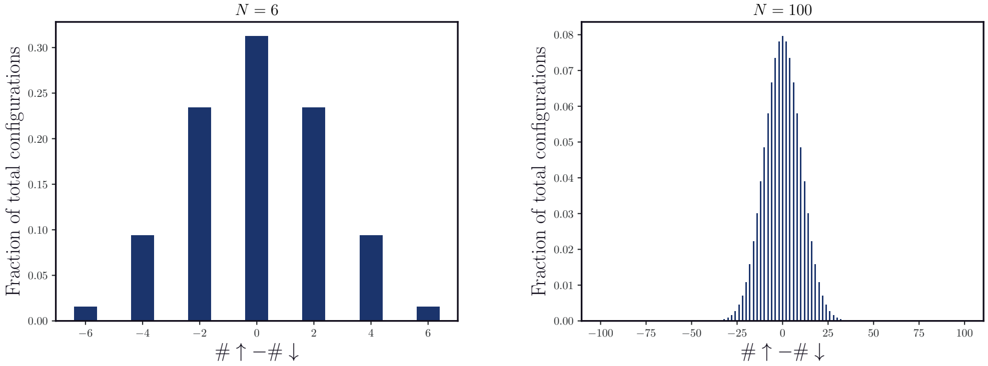


Figure 1.2: Ordered configurations are fewer in number, the most likely value of the order parameter in the absence of constraints is the one with maximal entropy.

we can count the number of states and make a histogram. As shown in Fig. 1.2, the typical configurations have as many up spins as down spins. As the system becomes large in size, substantial deviations from the mean value of our order parameter become profoundly rare. This is evident just by repeating the computation for $N = 100$.

We have introduced the notion of equally probable microstates, but why should this be the case? The answer is related to notions of *chaos*, or more precisely, a term called *ergodicity*. Ergodicity is a property of dynamics, the time evolution of a physical system. So far, we have not discussed this notion at all. So, how does a physical system evolve in time? To the best of our measurement abilities, the answer appears to be the time-dependent Schrödinger equation, but at scales larger than a single isolated molecule, we usually model this dynamics classically. At a high level, we can define a dynamics to be ergodic if, given any microstate $\nu_0$ a distinct microstate $\nu_1$ will be visited in some time $t < \infty$. When this is the case, every microstate is accessible and there is no external constraint to differentiate one configuration from the next. Mathematially, at fixed $(N, V, E)$,

$$P(\nu) = \begin{cases} \frac{1}{\Omega(N,V,E)} & \text{if } N(\nu) = N, V(\nu) = V, E(\nu) = E, \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

where

$$\Omega(N, V, E) = \sum_{\text{allowed } \nu} 1 = \text{\# allowed microstates.} \tag{1.4}$$

This function, $\Omega$, takes a thermodynamic state as an input and returns a number. For complex systems, for example, when $N$ is very large, $\Omega$ is an unfathomably large number. For the example we have already considered, a spin chain,

$$\Omega(N) = (\text{\# states for each spin })^N = 2^N. \tag{1.5}$$

This means that when $N = 100$, the probability that the system is observed in the all spin up configuration is $1/2^{100}$, very small indeed.

# Lecture 2 Boltzmann Statistics

## Recap

Last time, we introduced the microcanonical partition function. We discussed

1. Fluctuations are ubiquitous at small scales.
2. Ergodicity and chaos lead to a uniform distribution of microstates $\nu$ at fixed $N, V, E$.
3. We defined the *microcanonical partition function* $\Omega(N, V, E)$, which is the normalization constant for the probability distribution of microstates.

## 2.1 From counting to scaling

The normalization constant of a probability distribution over microstates is simply the sum of all the statistical weights, for every valid configuration in the ensemble. With the microcanonical postulate that the statistical weight of every configuration is 1, we obtain

$$\Omega(N, V, E) = \sum_{\text{allowed } \nu} w(\nu) = \sum_{\text{allowed } \nu} 1. \tag{2.1}$$

This, of course, means that computing the microcanonical partition function is just counting the number of allowed states.

For simple problems, this counting exercise is not so difficult. We saw last time that for a simple system of non-interacting binary spins, $\Omega(N) = 2^N$. Typically, molecular systems are much more complex than that, but we do have good strategies to help simplify the problem. A very general characteristic of the microcanonical partition function is that it grows rapidly with the number of particles $N$. Consider the system depicted in Fig. 2.1. We have divided space in this example into $M$ lattice cells. Let's call the volume of each subregion $v$ so that $V = Mv$. We can write

$$\rho = \text{density} = \frac{N}{Mv} \implies M = \frac{N}{\rho v}. \tag{2.2}$$

If $v$ is sufficiently large that the regions are statistically independent, which we assume to be the case, then we can write the total number of states as the product of the number of states within each cell. Denote

$$\tilde{\omega} = \text{single cell partition function} \tag{2.3}$$

hence,

$$\Omega = \tilde{\omega}^M = \left(\tilde{\omega}^{1/\rho v}\right)^N \equiv \omega^N. \tag{2.4}$$

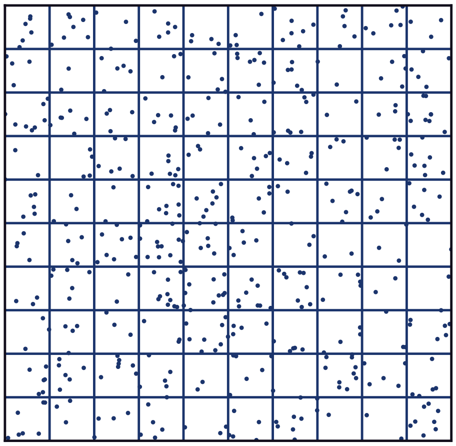


Figure 2.1: Partition a system in statistically independent regions with volume $v$ and then use the "grid" partition function to build the total partition function.

The function depends on the single cell variables $v, \rho, \epsilon$ where $\epsilon = E/V$ is an energy density. These variables are *intensive* meaning that that they do not grow as the system size grows. In contrast, $N$, $V$, and $E$ are *extensive*.

From (2.4), we see that

$$\log \Omega(N, V, E) \approx N \log \omega(\rho, v, \epsilon) \equiv I(\rho, v, \epsilon) \tag{2.5}$$

and hence,

$$p(\nu) \asymp e^{-NI(\rho,v,\epsilon)}. \tag{2.6}$$

The symbol $\asymp$ means that the two are asymptotically equivalent, so the assumption is that $N$ is very large. We call $I$ a large deviation rate function because it tells us the rate of decay of the probability of a configuration.

As we have discussed, most systems that we encounter in chemistry are not isolated and therefore do not satisfy the microcanonical postulate. Let's consider a system which is coupled to a large reservoir of energy, which we will refer to as a bath, and assume that the system and the bath together have fixed energy. That is, the system and the bath together are microcanonical. Suppose now that the system is adopts microstate $\nu$. The bath can then take on any state $\nu_B$ that satisfies,

$$E_{\mathrm{B}}(\nu_{\mathrm{B}}) = E_{\mathrm{T}} - E(\nu). \tag{2.7}$$

To compute the probability of the microstate $\nu$, we simply look at the joint probability of the system and the bath,

$$p(\nu) = \sum_{\nu_{\mathrm{B}},\ \text{valid}} p(\nu, \nu_{\mathrm{B}}) = \frac{1}{\Omega(E_{\mathrm{T}})} \Omega_{\mathrm{B}}(E_{\mathrm{T}} - E(\nu)). \tag{2.8}$$

This indicates that the probability of the microstate $\nu$ is directly proportional to the number of states of the bath.

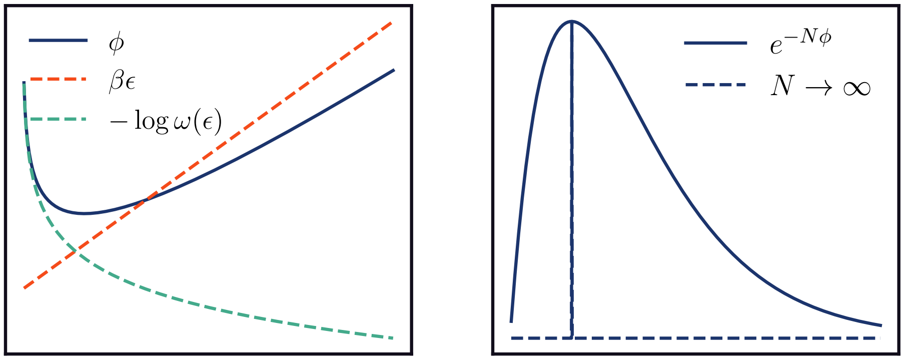


Figure 2.2: The large deviation form of the energy distribution reveals statistical concentration near the most likely energy in the thermodynamic $N \to \infty$ limit.

We are in a regime where the energy $E(\nu)$ is much smaller than the total energy in the system. That means that we can view $\Omega_\mathrm{B}(E_\mathrm{T} - E(\nu))$ as just a small perturbation away from $E_\mathrm{T}$. In this situation, one is inclined to Taylor expand; here we will Taylor expand the $\log \Omega_\mathrm{B}$ because these functions are growing rapidly and the log is a monotonic function that we can invert after the expansion. This yields

$$\log \Omega_\mathrm{B}(E_\mathrm{T} - E) = \log \Omega_\mathrm{B}(E_\mathrm{T}) + (E_\mathrm{T} - E)\left(\frac{\partial \log \Omega_\mathrm{B}}{\partial E_\mathrm{B}}\right)_{N_\mathrm{B}, V_\mathrm{B}} + \mathcal{O}(E^2) \tag{2.9}$$

The partial derivative that we have written, which holds $N_\mathrm{B}$ and $V_\mathrm{B}$ fixed, has a thermodynamic identity

$$\left(\frac{\partial \log \Omega_\mathrm{B}}{\partial E_\mathrm{B}}\right)_{N_\mathrm{B}, V_\mathrm{B}} \equiv \beta, \tag{2.10}$$

which for now we will simply think of as a property of the bath. Putting the pieces together this calculation tells us that

$$p(\nu) \propto e^{-\beta E(\nu)}, \tag{2.11}$$

which is the distribution known as the Boltzmann distribution; going from the expansion to this form, we drop all the constant terms that do not depend on $\nu$. The quantity $\beta = \frac{1}{k_\mathrm{B}T}$.

Many microstates share the same energy, though. To compute $p(E)$, we need to consider all microstates with energy $E$, of which there are $\Omega(E)$.

$$P(E) = \sum_{\nu \text{ with } E(\nu)=E} e^{-\beta E} = e^{-\beta E}\Omega(E). \tag{2.12}$$

In large deviation form, we can write this as

$$e^{-\beta N\epsilon + N\log\omega(\epsilon)} \equiv e^{-N\phi}. \tag{2.13}$$

Fig. 2.2 reveals the behavior of this function as $N$ becomes large. The distribution of energy concentrates on a single value and fluctuations away from this value are exponentially suppressed. When $N$ becomes comparable to a macroscopic number of particles, e.g. $N \sim 10^{24}$, the extent of this concentration is overwhelming and only the dominant configurations remain. This limit is known as the thermodynamic limit.

## 2.2 The other entropy

Now, let us return to $\log \Omega$. This quantity is proportional to the entropy and, of course, it's an important quantity in thermodynamics. In particular,

$$S(N, V, E) = k_{\mathrm{B}} \log \Omega(N, V, E) \tag{2.14}$$

where $k_{\mathrm{B}}$ is Boltzmann's constant.

Let's think about this in a simple $k$-level system with $N$ total particles. We will specify the occupancy of the $i$th energy level as $n_i$ and the energy can be directly computed from the occupancy statistics. The total energy of the system must satisfy

$$E = \epsilon_0 n_0 + \epsilon_1 n_1 + \cdots + \epsilon_{k-1} n_{k-1} \tag{2.15}$$

In this system, the total number of states can be specified by choice functions for the individual energy levels:

$$\Omega = \frac{N!}{(N-n_0)!n_0!} \times \frac{(N-n_0)!}{(N-n_0-n_1)!n_1!} \times \dots \tag{2.16}$$

after the telescopic cancelations, we are left with

$$\Omega = \frac{N!}{n_0!n_1!\dots n_{k-1}!} \tag{2.17}$$

The typical value of the occupancy for each level $n_i$ differs with energy (there are lower populations of the high energy states because they may violate the constraint that the total energy is fixed). This in mind, we can write

$$\Omega(N, V, E) = \frac{N!}{\prod_{i=0}^{k-1} (p(n_i)N)!} \tag{2.18}$$

where $p(n_i)$ is the probability of finding a particle in the $i$th energy level.

Because $N$ is very large, $N!$ is unfathomably large. Let's take logarithms and use Stirling's approximation

$$\log N! \approx N \log N - N \tag{2.19}$$

so that

$$\begin{aligned} \log \Omega &\approx N \log N - N - \sum_i p(n_i) N \log\left(p(n_i)N\right) - p(n_i)N \\ &= -N \sum_i p(n_i) \log p(n_i) \end{aligned} \tag{2.20}$$

This is another definition of the entropy, the one due to Gibbs,

$$k_{\mathrm{B}} \log \Omega = -k_{\mathrm{B}} N \sum_{i=0}^{k} p(n_i) \log p(n_i). \tag{2.21}$$

Rescaling by a factor of $N$, this definition coincides with the information theoretic definition of the entropy. Conceptually, this quantity is an average of $\log p$, so if, for example, $\log p(n_0) = 1$ and all particles are in the ground state, then there is zero entropy. Which distribution maximizes the entropy?

# Lecture 3 Connection to statistical information

## Recap

1. We derived the Boltzmann distribution,
$$p(\nu) \propto e^{-\beta E(\nu)}, \tag{3.1}$$
where $\beta = \frac{1}{k_\mathrm{B}T}$ is the inverse temperature in units of energy (i.e., $k_\mathrm{B}$ has units of energy per unit temperature). We obtained this form by Taylor expanding around the microcanonical partition function.
2. We defined the Boltzmann entropy
$$S(N, V, E) = k_\mathrm{B} \log \Omega(N, V, E). \tag{3.2}$$
3. We used Stirling's approximation,
$$\log N! \approx N \log N - N, \tag{3.3}$$
to show that
$$S(N, V, E) = -N k_\mathrm{B} \sum_\nu p(\nu) \log p(\nu). \tag{3.4}$$

## A new ensemble

We derived the Boltzmann distribution by relaxing the constraint that the system we study has constant energy. To this point, we have written the Boltzmann distribution as a proportionality, which only allows to compare the relative probability of specific configurations of a system. To give a proper probability, we must normalize the distribution,
$$p(\nu) = \frac{e^{-\beta E(\nu)}}{\sum_\nu e^{-\beta E(\nu)}}. \tag{3.5}$$
The denominator of this expression, which is just a number when $\beta$ is fixed at some particular value, we will denote
$$Z(\beta) := \sum_\nu e^{-\beta E(\nu)}. \tag{3.6}$$
Mathematically, $Z$ is called a normalization constant. The statistical mechanical name for $Z$ is the *canonical partition function*. Last time, we showed that as $N \to \infty$, the distribution concentrates on a single value of the energy, which was the average energy $E_*$. This means in the thermodynamic limit, the $N, V, E_*$ ensemble and the $N, V, T$ ensemble coincide, a phenomenon known as *equivalence of ensembles*.

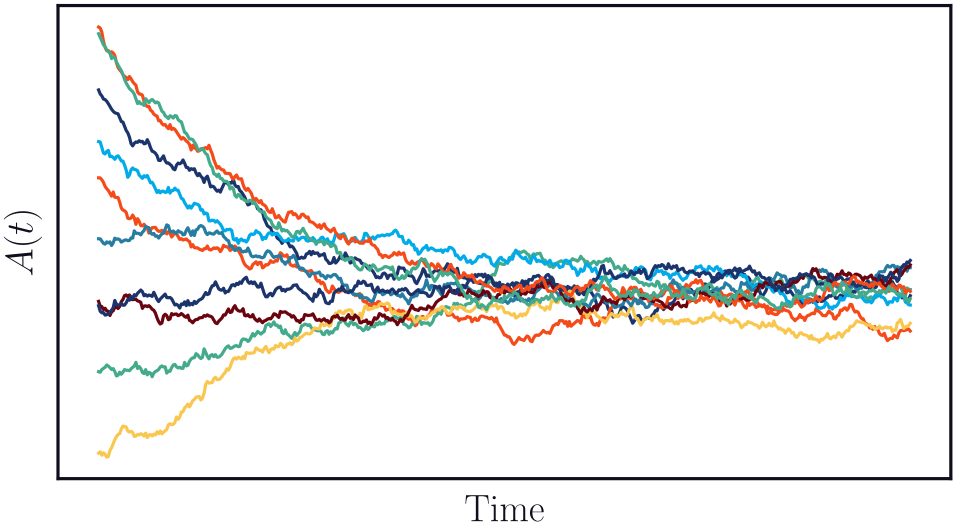


Figure 3.1: Initial conditions are forgotten in sufficiently long trajectories. We can then exchange the time average and ensemble averages.

## Thinking dynamically

We have already mentioned the concept of an ergodic dynamics, which we defined to be one in which every microstate will eventually be visited, regardless of the initial condition of the system. In statistical mechanics textbooks, you will often encounter a different definition of ergodicity: that the time average and the ensemble average are interchangeable. That is, for any observable $A$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T A(t)dt = \sum_\nu A(\nu)p(\nu). \tag{3.7}$$

Our central goal in statistical mechanics is to compute physical properties, i.e., averages and fluctuations of physical observables, by evaluating $p(\nu)$.

## Getting information from the distribution

We must now ask ourselves, what information about the system and its properties does $p(\nu)$ actually contain? If we fix the thermodynamic state $N, V, T$, then we know that the Boltzmann distribution governs the system. When this is the case, we can compute the energy distribution for a large system as follows. First,

$$Z(\beta) = \sum_\nu e^{-\beta E(\nu)} = \sum_E e^{-\beta E}\Omega(E). \tag{3.8}$$

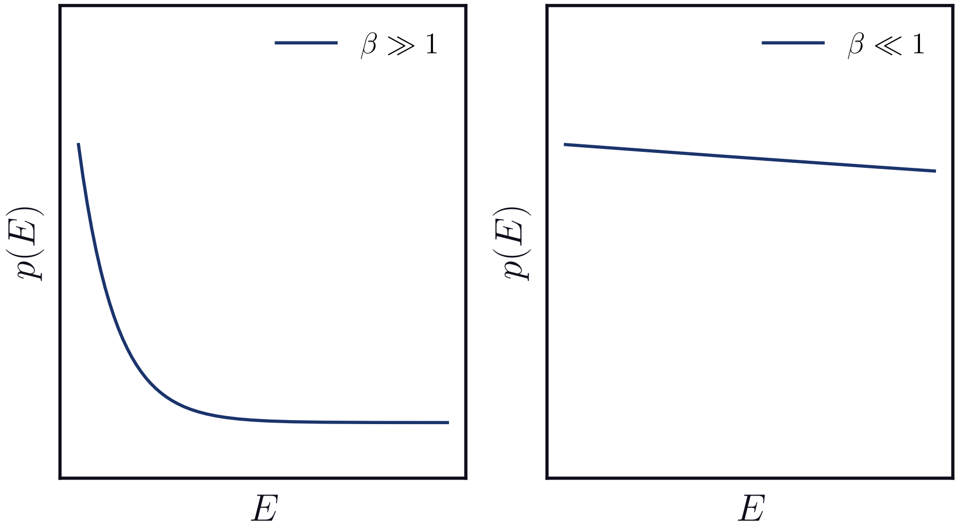


Figure 3.2: At low temperatures, the distribution of energies decays away from the ground state. At higher temperatures, the distribution of states becomes increasingly uniform.

When $N \gg 1$, the spacing between energy levels is extremely small, meaning that we can convert the sum to its continuous representation, the integral. Hence, for sufficiently large $N$,

$$Z(\beta) = \int \mathrm{d}E e^{-\beta E} \Omega(E) = \int \mathrm{d}\epsilon \exp\left(-N\left[\beta\epsilon - \log\omega(\epsilon)\right]\right). \tag{3.9}$$

As we observed last time, the value of $\epsilon$ that minimizes the exponent of the exponential function dominates the integral. We typically call this a saddle-point approximation. That in mind, we know that it must be the case that the function

$$a(\epsilon) = \beta\epsilon - \log\omega(\epsilon) \tag{3.10}$$

will satisfy

$$\left(\frac{\partial a}{\partial \epsilon}\right)\bigg|_{\epsilon_*} = 0. \tag{3.11}$$

In turn, this implies that

$$\beta = \left(\frac{\partial \log\omega}{\partial\epsilon}\right)\bigg|_{\epsilon_*} \iff \frac{1}{k_{\mathrm{B}} T_{\mathrm{bath}}} = \left(\frac{\partial S}{\partial E}\right)_{N,V} . \bigg|_{N\epsilon_*} \tag{3.12}$$

which gives a natural definition for the temperature of the system.

What is more, when we let $N$ tend towards infinity, the thermodynamic limit, we see

$$\begin{aligned} Z(\beta) &= \exp\left(-N\left[a(\epsilon_*)\right]\right) \\ &= \exp\left(-N\left[\beta\epsilon_* - \log\omega(\epsilon_*)\right]\right) \\ &= \exp\left(-\beta\left[E_* - TS(E_*)\right]\right) \equiv e^{-\beta A} \end{aligned} \tag{3.13}$$

In this expression,

$$A = E - TS \tag{3.14}$$

which is of course the Helmholtz free energy. This free energy $A$ is a natural function of $N$, $V$, and $T$.

We now have a way of translating between a thermodynamic function and a purely statistical mechanical object. Namely, we have derived the following relation,

$$A = -\beta^{-1} \log Z(\beta). \tag{3.15}$$

In thermodynamics, we extract information about a system by using differential relations and the derivatives of log $Z$ will be similarly informative. For example,

$$\begin{aligned} \frac{\partial(\beta A)}{\partial\beta} &= -\frac{\partial}{\partial\beta}\left(\log\sum_\nu e^{-\beta E(\nu)}\right) \\ &= \frac{\sum_\nu E(\nu)e^{-\beta E(\nu)}}{\sum_\nu e^{-\beta E(\nu)}} \equiv \langle E\rangle \end{aligned} \tag{3.16}$$

Of course, the probability distribution over microstates contains much more information than this.

$$\begin{aligned} \frac{\partial\langle E\rangle}{\partial\beta} &= -\frac{\partial^2}{\partial\beta^2}\left(\log\sum_\nu e^{-\beta E(\nu)}\right) \\ &= -\frac{\partial}{\partial\beta}\frac{\sum_\nu E(\nu)e^{-\beta E(\nu)}}{\sum_\nu e^{-\beta E(\nu)}} \\ &= \left\langle E^2\right\rangle - \langle E\rangle^2 \end{aligned} \tag{3.17}$$

Something remarkable has just occurred. It is no coincidence.

By taking derivatives of the logarithm of the partition function (which we showed coincides with the Helmholtz free energy in the thermodynamic limit), we obtained expressions for average and the fluctuations in energy. These statistical objects, the mean, the variance, and so on, are called cumulants and log $Z(\beta)$ is a cumulant generating function for the energy. The derivative that we computed in (3.17) has a clear physical meaning: there is an equivalence between the fluctuations in energy around the equilibrium and the magnitude of a response to changes in the parameter $\beta$.

## Lecture 4 Thermodynamic laws

## Recap

1. We defined the canonical partition function
$$Z(\beta) = \sum_\nu e^{-\beta E(\nu)}.$$
2. We related $Z(\beta)$, a purely statistical mechanical object, to a thermodynamic quantity, the Helmholtz free energy,
$$A(N, V, T) = -\beta^{-1} \log Z(\beta).$$
3. We took derivatives of $\log Z(\beta)$ to compute statistical properties of the energy distribution,
$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z(\beta),$$
and
$$\mathrm{var}(E) = \left\langle E^2 \right\rangle - \langle E \rangle^2 = -\frac{\partial^2}{\partial \beta^2} \log Z(\beta).$$

## Cumulants, cumulant generating functions

Probability distributions can be characterized by their *moments*,

$$\mu_n = \int_D x^n p(x) dx \equiv \langle x^n \rangle .$$

Cumulants are a closely related set of quantities, and they arise in many different contexts. The first few have names: the "mean", "variance", and "skewness" of a distribution.

| Cumulant | In terms of moments | Explicitly |
|---|---|---|
| $\kappa_1$ | $\mu_1$ | $\langle x \rangle$ |
| $\kappa_2$ | $\mu_2 - \mu_1^2$ | $\left\langle x^2 \right\rangle - \langle x \rangle^2$ |
| $\kappa_3$ | $\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$ | $\left\langle x^3 \right\rangle - 3 \left\langle x^2 \right\rangle \langle x \rangle + \langle x \rangle^3$ |
| $\vdots$ | | |

These formulae come from a rather simple expression, namely, they are associated with the generating function

$$g(\lambda) = \log \left\langle e^{\lambda x} \right\rangle \tag{4.1}$$

and

$$\kappa_n = \left. \frac{\mathrm{d}g}{\mathrm{d}\lambda} \right|_{\lambda=0} . \tag{4.2}$$

The structure of the cumulant generating function is, up to some constants, *exactly* the same as the structure of our free energy $A = -\beta^{-1} \log Z(\beta)$. That is,

$$Z(\beta) = \sum_{\nu} e^{-\beta E(\nu)}, \tag{4.3}$$

which we can think of as a microcanonical average, so that

$$\log Z(\beta) = \log \left\langle e^{-\beta E} \right\rangle \tag{4.4}$$

and hence derivatives of $\log Z$ with respect to $-\beta$ give information about the fluctuations in the energy.

As we emphasized last time, there is salient physical information contained in the fluctuations. Let's suppose, for example, we want to compute the heat capacity at constant volume,

$$C_V(T) = \left( \frac{\partial \langle E \rangle}{\partial T} \right)_{N,V} \tag{4.5}$$

using the relation

$$\frac{\partial \beta}{\partial T} = -\frac{1}{k_{\mathrm{B}} T^2} \tag{4.6}$$

we see that

$$-\left( \frac{\partial \langle E \rangle}{\partial \beta} \right)_{N,V} \frac{\partial \beta}{\partial T} = C_V(T). \tag{4.7}$$

We conclude that

$$k_{\mathrm{B}} T^2 C_V(T) = \mathrm{var}(E). \tag{4.8}$$

This fluctuation-response relation encodes how microscopic fluctuations influence a macroscopic thermodynamic property.

## Condensed Thermodynamics

We are going to very quickly recollect "all" of thermodynamics[*]. We will use many ideas and concepts from thermodynamics throughout this course.

Energy can neither be created nor destroyed; all changes in energy must be ascribed to either work or heat flow. Mathematically, we write

$$dE = đ\, \mathcal{Q} + đ\, \mathcal{W}. \tag{4.9}$$

---

[*] At least, you will have the basic ingredients to derive any thermodynamic relation you might ever need.

We use the symbol $\text{đ}$ to denote the fact that work and heat are not *state functions*. While this is a thermodynamic concept, it is equally valid at the microscopic scale. For example, let us consider the following experiment: A single mRNA molecule is conjugated to a polystyrene bead at both the 3' and 5' ends. A micropipette is used to stabilize one end, and the other is optically trapped using a laser.

Suppose beforehand we determined with a bulk measurement that the typical length of the mRNA is $\ell_*$. We do work on the molecule by pulling,

$$\text{đ}\, W = f_{\text{trap}} \mathrm{d}\ell. \tag{4.10}$$

Intuitively, let us imagine doing this process both rapidly and very slowly. In the fast process, we will apply a large force per unit length to unfurl the molecule. In the slow process, this occurs more gently, allowing the system to re-equilibrate as we pull. That is, we expect

$$\int_{\text{fast}} \text{đ}\, \mathcal{W} > \int_{\text{slow}} \text{đ}\, \mathcal{W}. \tag{4.11}$$

After running our protocol, we let the system come to equilibrium with a constraint, $\ell = \ell_{\text{f}}$. Unlike the work, the total energy of the molecule is a state function, meaning that

$$\int_{\text{fast}} \mathrm{d}E = \int_{\text{slow}} \mathrm{d}E = \Delta E. \tag{4.12}$$

In other words, it must be the case that

$$\int_{\text{fast}} \text{đ}\, \mathcal{Q} < \int_{\text{slow}} \text{đ}\, \mathcal{Q}. \tag{4.13}$$

We said that $\ell_*$ is the equilibrium length, by which we meant that it was the average value for an ensemble measurement. We know intuitively that when we release the constraint, $\ell = \ell_{\text{f}}$, the system will relax back to having a typical length $\ell_*$. To understand how the system arrives at this equilibrium configuration, lets do a simple thought experiment. The internal energy of the mRNA molecule is a complicated function, and configurations with a fixed energy $\epsilon$ have different lengths. So, let us imagine selecting the sub-ensemble of molecules with length $\ell_* + \delta\ell$

$$S(\epsilon, \ell) \geq S(\epsilon; \text{constraint } \ell = \ell_* + \delta\ell). \tag{4.14}$$

This observation is variational statement of *The Second Law of Thermodynamics*. There are many distinct formulations, but perhaps the most general is simply

> The change in entropy for a spontaneous process is positive.

Importantly, this is only true at the ensemble level. Fluctuations in microscopic systems can transiently violate the Second Law, but will never do so on average.

In Thermodynamics, we typically work with differentials to do calculations. For example, the total differential of the entropy is

$$\mathrm{d}S = \left(\frac{\partial S}{\partial E}\right)_{\boldsymbol{X}} \mathrm{d}E + \left(\frac{\partial S}{\partial \boldsymbol{X}}\right)_{E} \cdot \mathrm{d}\boldsymbol{X}, \tag{4.15}$$

where $\boldsymbol{X}$ is a vector of mechanical parameters. We also know that, for a reversible process

$$\mathrm{d}E = đ\,\mathcal{Q}_{\text{rev}} + f \cdot \mathrm{d}\boldsymbol{X}, \tag{4.16}$$

which is just a restatement of the First Law. If the process is also adiabatic, meaning that there is no net heat flow into the system, then

$$\mathrm{d}E = \mathrm{d}W = f \cdot \mathrm{d}\boldsymbol{X}, \tag{4.17}$$

noting that in this case the differential for the work is an *exact* differential. But it is also the case that the change in entropy for a reversible, adiabatic process must be zero, so

$$\mathrm{d}S = 0 \implies -f/T \cdot \mathrm{d}\boldsymbol{X} = \left(\frac{\partial S}{\partial X}\right)_E \cdot \mathrm{d}\boldsymbol{X}. \tag{4.18}$$

This means that entropy can act as a bona-fide force! It is a statistical force, but a force nontheless. Hence, we have obtained the relation

$$\mathrm{d}S = \frac{1}{T}\mathrm{d}E - \frac{f}{T} \cdot \mathrm{d}\boldsymbol{X}. \tag{4.19}$$

Alternatively,

$$\mathrm{d}E = T\mathrm{d}S + f \cdot \mathrm{d}\boldsymbol{X}. \tag{4.20}$$

For any process, we can write,

$$\mathrm{d}E = đ\,\mathcal{W}_{\text{rev}} + đ\,\mathcal{Q}_{\text{rev}} = T\mathrm{d}S + đ\,\mathcal{W} \tag{4.21}$$

which implies

$$đ\,\mathcal{W} \geq đ\,\mathcal{W}_{\text{rev}} \tag{4.22}$$

since

$$\mathrm{d}S \geq đ\,\mathcal{Q}/T, \tag{4.23}$$

which is the Clausius Inequality, yet another statement of the Second Law. Equivalently,

$$\mathrm{d}S \geq \frac{1}{T}(\mathrm{d}E - đ\,W) \implies đ\,W \geq \mathrm{d}(E - TS) = \mathrm{d}A \tag{4.24}$$

when $T$ is fixed. Yet, again, we end with Helmholtz! This statement says that the work that can be extracted from the system reversibly is exactly the *free energy*.

# Lecture 5 Connecting thermodynamics to statistical mechanics

## Recap

In the last lecture, we revisited ideas from Thermodynamics.

1. We stated the First Law
$$dE = \text{đ}\mathcal{W} + \text{đ}\mathcal{Q}, \tag{5.1}$$
which asserts that the change in energy must be attributable to work and heat exchange. The work we wrote
$$\text{đ}\mathcal{W} = \vec{f} \cdot d\boldsymbol{X} \tag{5.2}$$
where $\boldsymbol{X}$ consists of a collection of mechanical parameters (conventionally, $V$ and $N$) and $\vec{f}$ is a vector of conjugate forces. This is a central physical postulate of thermodynamics and it holds independent of the size of the system.

2. We discussed the Second Law, which we stated variationally. The implications were that
$$\Delta S \geq 0 \tag{5.3}$$
for any spontaneous process. Furthermore, using both of the above relations, we obtained the Clausius Inequality,
$$dS \geq \frac{\text{đ}\mathcal{Q}}{T}. \tag{5.4}$$

## 5.1 Mathematical framework of Thermodynamics

In the previous lecture, we wrote

$$\begin{aligned} dS &= \left(\frac{\partial S}{\partial E}\right)_{\boldsymbol{X}} dE - \left(\frac{\partial S}{\partial \boldsymbol{X}}\right)_{E} \cdot d\boldsymbol{X}, \\ &= \frac{1}{T} dE - \frac{\vec{f}}{T} \cdot d\boldsymbol{X}. \end{aligned} \tag{5.5}$$

Combining this expression with the First Law, and letting $\boldsymbol{X} = (V, N)$, we see

$$dE = TdS - pdV + \mu dN. \tag{5.6}$$

Does this mean that the equality

$$E(S, V, N) = TS - pV + \mu N \tag{5.7}$$

holds?

In fact, it does, but that is not necessarily the case for an arbitrary function. The energy is *extensive*, meaning that if we increase the size of the system, the energy increases proportionally. Mathematically, we write

$$E(\lambda S, \lambda V, \lambda N) = \lambda E(S, V, N). \tag{5.8}$$

Functions with this property are called homogeneous of degree one—or, more simply, linearly extensive. Because

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\lambda S, \lambda V, \lambda N) = \left(\frac{\partial E}{\partial S}\right)_{N,V}\frac{\mathrm{d}\lambda S}{\mathrm{d}\lambda} + \left(\frac{\partial E}{\partial V}\right)_{N,S}\frac{\mathrm{d}\lambda V}{\mathrm{d}\lambda} + \left(\frac{\partial E}{\partial N}\right)_{S,V}\frac{\mathrm{d}\lambda N}{\mathrm{d}\lambda}, \tag{5.9}$$

and all the extensive variables that appear are linear in $\lambda$ (i.e., $\frac{\mathrm{d}\lambda S}{\mathrm{d}\lambda} = S$),

$$\boxed{E = TS - pV + \mu N. \tag{5.10}}$$

The resulting expression is a *thermodynamic potential*. It is a natural function of $S$, $V$, and $N$. We can easily change that dependence, though, simply by subtracting conjugate pairs. For example,

$$E - \left(\frac{\partial E}{\partial S}\right)_{N,V} S = E - TS \equiv A, \tag{5.11}$$

the Helmholtz free energy, which we already know depends on $T$ as opposed to $S$. Its total differential makes this clear:

$$dA = dE - SdT - TdS = -SdT - pdV + \mu dN \tag{5.12}$$

$$\boxed{dA = -SdT - pdV + \mu dN \tag{5.13}}$$

Similarly,

$$E - \left(\frac{\partial E}{\partial S}\right)_{N,V} S - \left(\frac{\partial E}{\partial V}\right)_{N,S} V = A + pV \equiv G, \tag{5.14}$$

the Gibbs free energy, which we know is a natural function of $N, p, T$. Again, the differential makes this dependence clear,

$$\boxed{dG = -SdT + Vdp + \mu dN \tag{5.15}}$$

The transformation that we are carrying out is known as a Legendre Transform. We will see them again.

## 5.2 Extracting Thermodynamics from Single Particle Partition functions

For a generic, complex system we cannot easily compute the microcanonical partition function $\Omega(N, V, E)$ nor the canonical partition function $Z(N, V, T)$. However, for some models we can

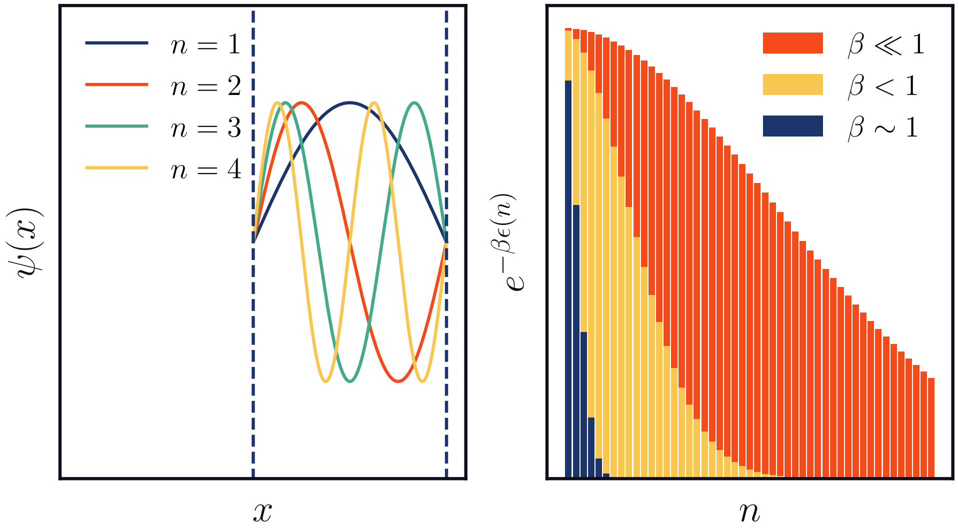


Figure 5.1: The decay of $e^{-\beta E}$ at high temperatures is small relative to integer spacing, meaning that we can approximate the sum with an integral.

carry out this computation explicitly, and it happens to be an extraordinarily useful exercise, because we will be able to build up a molecular partition function.

If we go back to the simplest systems, for example, an isolated particle spatially localized to some region of space, we can use basic quantum mechanical models of the energy. For example, let us take the particle in a box energy eigenstates depicted in Fig. 5.1. Recalling that the energies associated with the eigenstates are

$$\epsilon_{n_x} = \frac{n_x^2 h^2}{8ml_x^2}. \tag{5.16}$$

When the temperature is low, the spacing between the energies is large relative to $k_\mathrm{B}T$ and the function changes rapidly with $n$. At higher temperatures, the Boltzmann factor plotted in Fig. 5.1 changes gradually. When this is the case, we can replace

$$z_{\text{trans}}(\beta) = \sum_{n_x=1}^{\infty} \exp\left(-\beta \frac{n_x^2 h^2}{8ml_x^2}\right) \tag{5.17}$$

with

$$z_{\text{trans}}(\beta) \approx \int_0^{\infty} \exp\left(-\beta \frac{n_x^2 h^2}{8ml_x^2}\right) dn_x. \tag{5.18}$$

We immediately recognize this integral as a Gaussian integral and the solution is

$$z_{\text{trans}}(\beta) = \frac{\sqrt{2\pi m k_\mathrm{B} T}}{h} l_x. \tag{5.19}$$

In chemistry, we typically deal with three-dimensional systems, but fortunately, there is not much more work to do in order to treat that case. First, we recall that the particle in the box energies are given by

$$\epsilon_{n_x,n_y,n_z} = \frac{h^2}{8m}\left(\left(\frac{n_x}{l_x}\right)^2 + \left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2\right). \tag{5.20}$$

While this problem initially looks more complicated, because the $x$, $y$, and $z$ directions are independent, the associated partition function is *factorizable*. That is,

$$\begin{aligned} z_{\text{trans,3D}}(\beta) &= \sum_{n_x,n_y,n_z=1}^{\infty} e^{-\beta\epsilon(n_x,n_y,n_x)} \\ &= \left(\sum_{n}^{\infty} e^{-\beta\epsilon(n)}\right)^3. \end{aligned} \tag{5.21}$$

Therefore, the result we obtained in the 1D case suffices. In three spatial dimensions, the translational partition function is

$$z_{\text{trans,3D}}(\beta) = \left(\frac{2\pi m k_{\mathrm{B}} T}{h^2}\right)^{3/2} l_x l_y l_z \equiv \frac{V}{\Lambda^3}. \tag{5.22}$$

We can think of this expression as specifying the states available to the system with a resolution that is related to the temperature. Interestingly, this resolution is exactly the thermal de Broglie wavelength. In this sense, the quantum mechanical extent of a particle constrains our ability to localize it.

# Lecture 6 Molecular partition functions

## Recap

Last time, we focused on relating the various thermodynamic potentials, which we know can be obtained by taking the thermodynamic limit of statistical mechanical partition functions. We then narrowed the focus to minimal molecular system.

1. We first wrote

   $$E = TS - pV + \mu N \tag{6.1}$$

   or, equivalently (due to Euler's theorem for homogeneous functions),

   $$dE = TdS - pdV + \mu dN. \tag{6.2}$$

2. We showed that other thermodynamic potentials could be obtained by subtracting conjugate pairs (Legendre Transform)[*]

   $$A = E - \left(\frac{\partial E}{\partial S}\right)_{N,V} S \tag{6.3}$$

3. We computed the single particle translational partition function

   $$z_{\text{trans}}(\beta) = \frac{V}{\Lambda^3} \tag{6.4}$$

   where

   $$\Lambda = \frac{\sqrt{2\pi m k_{\mathrm{B}} T}}{h} \tag{6.5}$$

   is the thermal de Broglie wavelength.

## Building a molecular partition function

Computing a molecular partition function requires that we incorporate information about all the physical contributions to the energy. From elementary quantum mechanics, we know precisely what these contributions are:

$$\epsilon_{\text{mol}}^{(i,j,k,l)} = \epsilon_{\text{trans}}^{(i)} + \epsilon_{\text{vib}}^{(j)} + \epsilon_{\text{rot}}^{(k)} + \epsilon_{\text{elec}}^{(l)}. \tag{6.6}$$

---

[*] We will revisit this in a few weeks.

Here, we are including contributions from translations, vibrations, rotations, and electronic states. There is, additionally, a nuclear contribution to the energy that we will neglect for the time being and discuss in detail in the following lecture. Hence, the molecular partition function is

$$z_{\text{mol}}(\beta) = \sum_{ijkl} \exp\left(-\beta(\epsilon_{\text{trans}}^{(i)} + \epsilon_{\text{vib}}^{(j)} + \epsilon_{\text{rot}}^{(k)} + \epsilon_{\text{elec}}^{(l)})\right). \tag{6.7}$$

This partition sum has the mathematically convenient property that the energy is factorizable. Therefore, we can treat it term by term,

$$\begin{aligned} z_{\text{mol}}(\beta) &= \sum_i e^{-\beta\epsilon_{\text{trans}}^{(i)}} \sum_j e^{-\beta\epsilon_{\text{vib}}^{(j)}} \sum_k e^{-\beta\epsilon_{\text{rot}}^{(k)}} \sum_l e^{-\beta\epsilon_{\text{elec}}^{(l)}} \\ &= z_{\text{trans}}\, z_{\text{vib}}\, z_{\text{rot}}\, z_{\text{elec}}. \end{aligned} \tag{6.8}$$

It is worth emphasizing at this point that, although we are taking a microscopic point of view, the partition function we are computing will ultimately contain useful macroscopic information,

$$\langle E\rangle = -N\left(\frac{\partial \log z_{\text{mol}}}{\partial\beta}\right), \tag{6.9}$$

at least when the interactions between molecules can be neglected.

## 6.1 Rotational partition function

Each of the contributions to the energy we have written above has a simple quantum mechanical description via a canonical model. Going term by term, if we consider a heteronuclear diatomic molecule, the energy levels are quantized with a coefficient that depends on the moment of inertia of the rotor

$$I = \mu r^2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{6.10}$$

which we express in terms of the bond radius and the reduced mass $\mu$. The energy levels are given by

$$\epsilon_l = \frac{\hbar^2}{2I} l(l+1) \quad l = 0, 1, 2, \ldots \tag{6.11}$$

which we often write in terms of the rotational constant

$$B = \frac{h^2}{8\pi^2 I}. \tag{6.12}$$

To obtain a constant that can be directly compared to experimental rotational spectra, we typically write

$$\epsilon_l = \tilde{B}chl(l+1) \quad l = 0, 1, 2, \ldots \tag{6.13}$$

where

$$\tilde{B} = \frac{B}{ch} = \frac{h}{8\pi^2 Ic} \tag{6.14}$$

which has units of inverse length. Finally, to produce an expression for the partition function, we

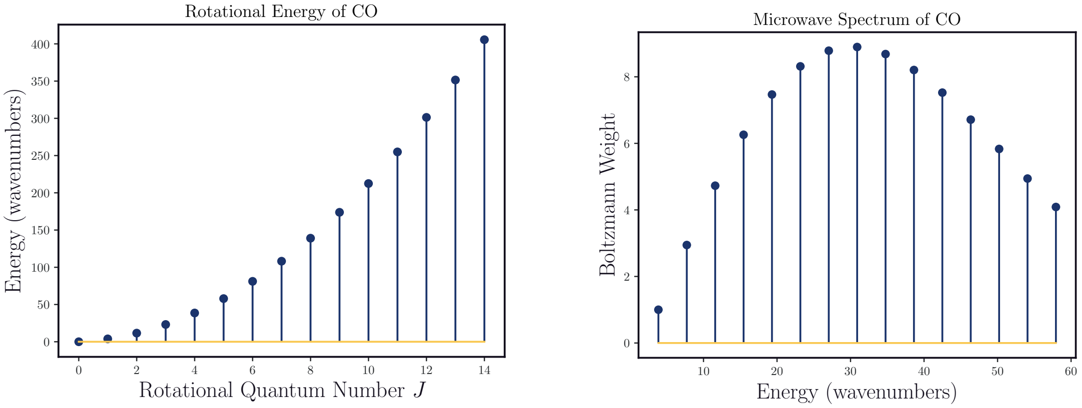


Figure 6.1: Spectral lines of CO.

must also recall that these energy levels are $g_l = 2l + 1$-fold degenerate. Explicitly,

$$\begin{aligned} z_{\text{rot}}(\beta) &= \sum_{l=0}^{\infty} g_l e^{-\beta \epsilon_{\text{rot}}^{(l)}} \\ &= \sum_{l=0}^{\infty} g_l e^{-\beta \tilde{B} ch l(l+1)}. \end{aligned} \tag{6.15}$$

The form of this sum looks familiar—the quadratic exponent suggests that we may be able to exploit a similar trick to the one we used to compute the translational partition function. To simplify this expression, we will define

$$\Theta_{\text{rot}} = \frac{\tilde{B} ch}{k_{\text{B}}} \tag{6.16}$$

and if $\Theta_{\text{rot}}/T \ll 1$, then approximating the sum by an integral should be a reasonable approximation.

So, in the high temperature regime, we have

$$z_{\text{rot}}(\beta) \approx \int_0^{\infty} (2l+1) e^{-\Theta_{\text{rot}} l(l+1)/T} dl. \tag{6.17}$$

This is now a moment of a Gaussian integral, so we have some tools to compute it. In this case, there is an elegant substitution, though, that is possibly even easier. Let

$$\begin{cases} u = l(l+1) \\ du = 2l+1 \end{cases} \tag{6.18}$$

and our integral becomes

$$\int_0^{\infty} e^{-\Theta_{\text{r}} u/T} du = \frac{T}{\Theta_{\text{r}}} \tag{6.19}$$

Using the definition of $B$, we have an explicit expression

$$z_{\text{rot}}(T) = \frac{8\pi^2 I k_{\text{B}} T}{h^2}. \tag{6.20}$$

For a simple heteronuclear diatomic, we can directly measure $\Theta_{\text{rot}}$ using microwave spectroscopy which excites in the 1–100 GHz range.

## 6.2 Vibrational partition function

The vibrational energy levels for a harmonic oscillator are

$$\epsilon_{\text{vib}}^{(n)} = \hbar\omega\left(n + \frac{1}{2}\right). \tag{6.21}$$

There is no degeneracy in these states, so we must compute

$$z_{\text{vib}}(\beta) = e^{-\beta\hbar\omega/2}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}. \tag{6.22}$$

This is a geometric series, which means we can compute the solution exactly:

$$z_{\text{vib}}(\beta) = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}. \tag{6.23}$$

In analogy with the expression we obtained for the rotational partition function, let us also define

$$\Theta_{\text{vib}} = \hbar\omega/k_{\text{B}}. \tag{6.24}$$

In the high temperature limit, we see that

$$z_{\text{vib}}(\beta) \approx \frac{T}{\Theta_{\text{vib}}}. \tag{6.25}$$

# Lecture 7 Microscopic derivation of the ideal gas law

## Recap

1. We wrote an expression for the rotational partition function in the high temperature limit:

$$z_{\mathrm{rot}}(\beta) = \frac{T}{\Theta_{\mathrm{rot}}}, \qquad \Theta_{\mathrm{rot}} = \frac{\tilde{B}ch}{k_{\mathrm{B}}} \tag{7.1}$$

   Importantly, we assumed that the molecule under consideration was heteronuclear. In the homonuclear case, we obtain the slightly modified expression,

$$z_{\mathrm{rot}}(\beta) = \frac{T}{2\Theta_{\mathrm{rot}}}. \tag{7.2}$$

2. We computed the exact expression for the vibrational partition function

$$z_{\mathrm{vib}}(\beta) = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}. \tag{7.3}$$

At this point, we have expression for all the ingredients of the molecular partition function except the electronic and nuclear components. We mentioned in the last lecture that these do not typically play a big role for chemistry in ambient conditions because the energy required to excite an electronic or nuclear transition is typically large relative to $k_{\mathrm{B}}T$. For completeness, to compute the electronic partition function, we sum over energy levels

$$z_{\mathrm{elec}}(\beta) = \sum_{l} g_l e^{-\beta\epsilon_{\mathrm{elec}}^{(l)}} \tag{7.4}$$

where $g_l$ denotes the degeneracy of the $l$th energy level.

A typical electronic excitation energy is on the order of 1eV. Because $k_{\mathrm{B}}T$ at 298K is 25.7meV, the relative probability of the first excited state to the ground state is low. To quantify "low", let's consider a relatively stable but somewhat typical example: monoatomic He. The first excited state has a degeneracy of 3 ($p$-orbitals) and the energy is approximately 20eV. This means that

$$p(\epsilon_{\mathrm{elec}}^{(1)}) = \frac{3e^{-\beta\epsilon_{\mathrm{elec}}^{(1)}}}{1 + 3e^{-\beta\epsilon_{\mathrm{elec}}^{(1)}} + \dots} \tag{7.5}$$

which is roughly

$$p(\epsilon_{\mathrm{elec}}^{(1)}) \approx 3e^{-20/0.025} \approx e^{-800}. \tag{7.6}$$

| partition function | model | approximation | constants |
|---|---|---|---|
| $z_{\text{trans}} = V/\Lambda^3$ | $\epsilon_{\text{trans}}^{(n)} = \frac{n^2h^2}{8ml^2}$ | high-temperature | $\Lambda = \frac{h}{\sqrt{2\pi m k_{\text{B}} T}}$ |
| $z_{\text{rot}} = \frac{T}{\sigma\Theta_{\text{rot}}}$ | $\epsilon_{\text{rot}}^{(l)} = \tilde{B}chl(l+1)$ | high-temperature | $\tilde{B} = \frac{\hbar^2}{2Ich} \quad \sigma = 1, 2$ * |
| $z_{\text{vib}} = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}$ | $\epsilon_{\text{vib}}^{(n)} = \hbar\omega(n + \frac{1}{2})$ | harmonic bonds | |
| $z_{\text{elec}} = g_0 e^{-\beta e_{\text{elec}}^{(0)}}$ | electronic ground state | $k_{\text{B}}T \ll 1\text{eV}$ | |
| $z_{\text{nuc}} = 2I + 1$ | ground state degeneracy | $k_{\text{B}}T \ll 10^6$ eV | $I$ nuclear spin quantum number |

Table 7.1: Summary of contributions to the single molecule partition function.

So, we can conclude that we do not see this state. That is, we can write

$$z_{\text{elec}}(\beta) = g_0 e^{-\beta\epsilon_{\text{elec}}^{(0)}} \tag{7.7}$$

In light of the fact that a typical nuclear transition requires a million eV, we can safely say that the nuclear contribution to the molecular partition function is dominated by the ground state degeneracy, which is related to the nuclear spin quantum number $I$,

$$z_{\text{nuc}}(\beta) = 2I + 1 \tag{7.8}$$

for natural conditions on Earth. Table 7.1 summarizes the components of the molecular partition function.

## 7.1 Thermodynamic information from the molecular partition function

For a sufficiently dilute system of $N$ molecules, we can write the total energy as

$$E(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) = \epsilon_{\text{mol}}(\boldsymbol{x}_1) + \ldots \epsilon_{\text{mol}}(\boldsymbol{x}_N). \tag{7.9}$$

This expression assumes that the particles are *non-interacting*, which is a reasonable approximation for a dilute gas. If there were inter-particle interactions, we would need to account for the interaction energies explicitly. Any time we have a separable energy function like this, we can factorize the associated partition function. This means that

$$Z(\beta) \propto \int_V e^{-\beta E(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)} d\boldsymbol{x}_1 d\boldsymbol{x}_2 \ldots d\boldsymbol{x}_N = \prod_{i=1}^{N} \int_V e^{-\beta\epsilon_{\text{mol}}(\boldsymbol{x})} d\boldsymbol{x}. \tag{7.10}$$

In other words, the partition function for the $N$-particle system should satisfy

$$Z(\beta) \propto z_{\text{mol}}^N(\beta). \tag{7.11}$$

The constant of proportionality is simply a counting factor. If the system consists of *indistinguishable particles*, then any permutation of the indices is an equivalent configuration—there is no physical difference and so the current expression over-counts.

$$Z(\beta) = \frac{1}{\text{\# permutations of } N} z_{\text{mol}}^N(\beta) = \frac{1}{N!} z_{\text{mol}}^N(\beta). \tag{7.12}$$

This expression is quite useful, and we can get a lot of thermodynamic information out of it. For example,

$$\begin{aligned}\langle E\rangle &= \left(\frac{\partial \log Z}{\partial(-\beta)}\right)_{N,V} \\ &= -N\left(\frac{\partial \log z_{\mathrm{mol}}}{\partial \beta}\right)_{N,V} \\ &= Nk_{\mathrm{B}}T^2\left(\frac{\partial \log z_{\mathrm{mol}}}{\partial T}\right)_{N,V}.\end{aligned} \tag{7.13}$$

Of course, because $\log Z$ is related to the Helmholtz free energy $A(N, V, T)$, we also have access to any information that we can extract from thermodynamic manipulations of $A$. Recalling that

$$dA = d(E - TS) = -SdT - pdV + \mu dN \tag{7.14}$$

we note that we could compute, for example, the pressure via

$$p = -\left(\frac{\partial A}{\partial V}\right)_{N,T}. \tag{7.15}$$

For a non-interacting monoatomic gas, the only contribution to this expression is translational energy. In other words,

$$\begin{aligned}p &= \beta^{-1}N\left(\frac{\partial \log V}{\partial V}\right)_{T,N} \\ &= \frac{N}{V}k_{\mathrm{B}}T\end{aligned} \tag{7.16}$$

which you will certainly recognize as the ideal gas law. This a fairly stunning calculation, despite its simplicity: we have deduced the ideal gas law purely from a quantum mechanical formulation of the energy, the assumption of independence, and the connection between statistical ensembles and thermodynamic potentials.

# Lecture 8 Equilibrium constants and partition functions

## Recap

1. Built a complete representation of the molecular partition function, including translational, vibrational, rotational, electronic, and nuclear contributions:
$$Z(\beta) = \frac{z_{\text{mol}}^N(\beta)}{N!}, \tag{8.1}$$
where
$$\begin{aligned} z_{\text{mol}} &= z_{\text{trans}} z_{\text{rot}} z_{\text{vib}} z_{\text{elec}} z_{\text{nuc}} \\ &= \frac{V}{\Lambda^3} \times \frac{T}{\sigma \Theta_{\text{rot}}} \times \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} \times g_0 e^{-\beta\epsilon^{(0)}_{\text{elec}}} \times (2I+1), \end{aligned} \tag{8.2}$$
a relatively simple expression considering how much information it contains.

2. We used this expression in the case of a monoatomic ideal gas to derive the ideal gas law
$$pV = Nk_{\text{B}}T \quad \text{or} \quad \beta p = \rho. \tag{8.3}$$

## A minimal view of equilibrium constants

What does chemical equilibrium mean from a thermodynamic perspective? We know that the entropy of a chemical process is related to the work done by the chemostats on the system. For a reaction involving $k$ species, we write
$$dS = -\sum_{i=1}^{k} \frac{\mu_i}{T} dN_i \tag{8.4}$$
which we could write in molar quantities using stoichiometric coefficients
$$dS = -\frac{d\bar{N}}{T} \sum_{i=1}^{k} \mu_i \nu_i \implies \sum_i \mu_i \nu_i = 0, \tag{8.5}$$
at equilibrium. That is, chemical equilibrium imposes a constraint on the chemical potentials. Note that the stability of all the thermodynamic potentials means that we could have written this relation in terms of the Helmholtz or Gibbs free energies, depending on the external conditions of the system.

We know that we can obtain information about chemical potentials from the various thermodynamic derivatives that coincide with the chemical potential, e.g. $\left(\frac{\partial G}{\partial N}\right)_{p,T}$. Because we have access to the Helmholtz free energy directly from the canonical partition function, we can also write the

Gibbs free energy in terms of the canonical partition function. For an ideal gas, the Legendre transform defining the Gibbs free energy $G$ is explicit,

$$A + pV = -k_{\mathrm{B}}T \log \frac{z^N}{N!} + Nk_{\mathrm{B}}T, \tag{8.6}$$

where we used the ideal gas law to express $pV$. This expression can be simplified using Stirling's approximation as

$$G = -Nk_{\mathrm{B}}T(\log \frac{ze}{N} - \log e) = -Nk_{\mathrm{B}}T \log \frac{z}{N}. \tag{8.7}$$

Equilibrium constants measure the way that mass *partitions* between products and reactants. We will first consider a very minimal model of a reaction. Consider the system illustrated in 8.1. In this system a state $B$ is converted into a state $A$. We measure the energies $\epsilon^A_{i,0}$ relative to the zero-point energy within each basin (i.e., there is not an *a priori* global reference). The reversible work associated with this conversion is $\Delta\epsilon_0$. If we assume $\Delta\epsilon_0 > 0$, then the reaction is endothermic because it absorbs heat from the environment to cross the barrier.

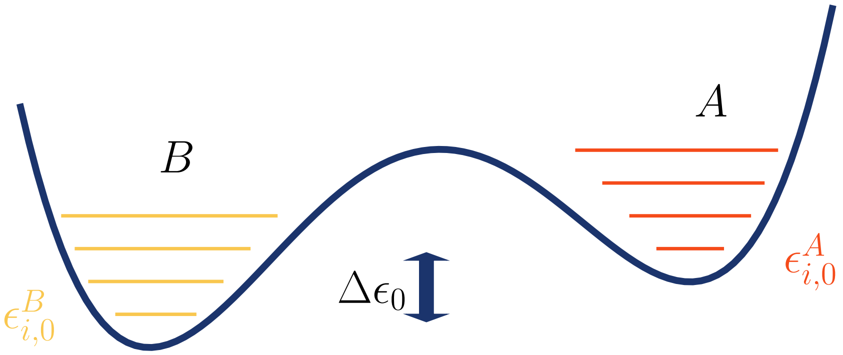


Figure 8.1: A simplistic view of a conversion of $B$ into $A$.

From considerations of statistical mechanics so far, we know that the probability of finding the system in $B$ or $A$ will be related to the energy associated with these respective states. To make a relative comparison, we can set $\epsilon^B_{0,0} = 0$ and shift all the energy levels by subtracting off this zero-point energy of $B$. We, of course, know how to compute the expected number of $A$ molecules, denoted $N_A$. This quantity is, up to normalization,

$$\langle N_A \rangle \propto N \sum_{i=0}^{\infty} e^{-\beta\epsilon^A_{i,0} - \beta\Delta\epsilon_0} = Nz_A e^{-\beta\Delta\epsilon_0}. \tag{8.8}$$

Similarly, the expected number of $B$ molecules is

$$\langle N_B \rangle \propto N \sum_{i=0}^{\infty} e^{-\beta\epsilon^B_{i,0}} = Nz_B. \tag{8.9}$$

Therefore, the partitioning of mass between the two species is given by a ratio of partition functions! It is just

$$\frac{\langle N_A \rangle}{\langle N_B \rangle} = \frac{z_A}{z_B} e^{-\beta \Delta \epsilon_0}, \tag{8.10}$$

noting the importance of the reference energy. This is our first statistical mechanical derivation of an equilibrium constant,

$$\boxed{K_{\text{eq}} = \frac{z_A}{z_B} e^{-\beta \Delta \epsilon_0}.} \tag{8.11}$$

Let's consider two different limits to ensure that the expression we have obtained is sensible. First, if $\beta \epsilon_0 \gg 1$, then $K_{\text{eq}} \approx 0$. In this regime, the ground state dominates and the system is almost entirely $B$ molecules. Second, if $\beta \epsilon_0 \approx 0$, then $K_{\text{eq}} \approx z_A / z_B$, the regime where the entropic effects contribute more.

## The "standard" formulation

We have promised that we would use the Gibbs free energy. From thermodynamics, we know that

$$G = A + pV = E - TS + pV \tag{8.12}$$

and, when holding $N$ fixed,

$$dG = -SdT + Vdp. \tag{8.13}$$

Working with molar quantities, for an ideal gas at constant temperature, constant number, we then have

$$dG = \frac{nRT}{p} dp. \tag{8.14}$$

Integrating this relation from $p_1$ to $p_2$ yields the familiar expression

$$\Delta G = G(p_2) - G(p_1) = nRT \log \frac{p_2}{p_1}. \tag{8.15}$$

When we talk about the free energy at a given pressure, we always have to worry about a relative shift,

$$G - G^{\circ} = RT \log \frac{p}{1\text{atm}} \tag{8.16}$$

which is the difference between the free energy and the standard state free energy (of one mole of gas at one atmosphere of pressure).

Now, if we consider a reaction

$$bB + cC \rightleftharpoons dD + fF, \tag{8.17}$$

the corresponding $\Delta G$ for the reaction can be written as

$$\Delta G = \sum_{i \in \text{species}} v_i \Delta G_i, \tag{8.18}$$

where $v_i$ are the stoichiometric coefficients, a fact that you likely have imprinted on somewhere deep within your brain. Using the expression for $\Delta G$ we wrote above, and that $\Delta G = 0$ at chemical equilibrium, we obtain

$$0 = \Delta G^{\ominus} + RT \log \frac{\left(\frac{p_D}{p^{\ominus}}\right)^d \left(\frac{p_F}{p^{\ominus}}\right)^f}{\left(\frac{p_B}{p^{\ominus}}\right)^b \left(\frac{p_C}{p^{\ominus}}\right)^c}. \tag{8.19}$$

In other words, the difference in the standard state free energy can be written in terms of the equilibrium constant:

$$\Delta G^{\ominus} = -RT \log \frac{\left(\frac{p_D}{p^{\ominus}}\right)^d \left(\frac{p_F}{p^{\ominus}}\right)^f}{\left(\frac{p_B}{p^{\ominus}}\right)^b \left(\frac{p_C}{p^{\ominus}}\right)^c} = -RT \log K_p. \tag{8.20}$$

So far this is basic chemistry, but the exciting thing is that we now know how to compute $G$ from the partition function. For species $j$, the *molar* Gibbs free energy at standard state is

$$G_j^{\ominus} = -RT \log \frac{z_j^{\ominus}}{N_A}, \tag{8.21}$$

where $N_A$ is now Avogadro's number. Correspondingly,

$$\begin{aligned} \Delta G^{\ominus} &= \sum_i v_i G_i^{\ominus} \\ &= -RT \log \prod_i \left(\frac{z_i^{\ominus}}{N_A}\right)^{v_i}. \end{aligned} \tag{8.22}$$

We now have a purely statistical mechanical expression for the equilibrium constant, which is

$$K_p = \prod_i \left(\frac{z_i^{\ominus}}{N_A}\right)^{v_i}. \tag{8.23}$$

Of course, just as before, we need to carefully adjust the energies relative to a global reference. If there is not a unique global reference, we have a factor

$$K_p = e^{-\Delta E_0/RT} \prod_i \left(\frac{z_{i,0}^{\ominus}}{N_A}\right)^{v_i}. \tag{8.24}$$

Here $z_{j,0}^{\ominus}$ refers to the partition function measured relative to the lowest energy in the mode $j$ and

$$\Delta E_0 = \sum_j v_j \epsilon_{j,0}. \tag{8.25}$$

# Lecture 9 Enumerating the ensembles

## Recap

1. Related $K_p$ to the partition function.
2. Expression we derived:

$$K_{\text{eq}} = e^{-\frac{\Delta E_0}{RT}} \prod_{i \in \text{species}} \left( \frac{z^{\ominus}_{i,0}}{N_A} \right)^{v_i} \tag{9.1}$$

where

$$\begin{aligned} \Delta E_0 &= \sum v_i \epsilon_{i,0} \\ v_i &= \text{stoichiometric coefficent} \\ z^{\ominus}_{i,0} &= \text{standard state partition function, energies relative to ground state of species } i \end{aligned} \tag{9.2}$$

## 9.1 Back to basics

Last time we established that

$$\Delta G^{\ominus} = -RT \log K_{\text{eq}}. \tag{9.3}$$

Thermodynamically, we know that the Gibbs free energy can be related to enthalpy, $H = E + pV$, using

$$dG = dH - TdS \tag{9.4}$$

which means we can write

$$\Delta G^{\ominus} = \Delta H^{\ominus} - T\Delta S^{\ominus} \tag{9.5}$$

or

$$\log K_{\text{eq}} = -\frac{\Delta H^{\ominus}}{RT} + \frac{\Delta S^{\ominus}}{R}. \tag{9.6}$$

Using the expression we previously derived for the equilibrium constant, we can interpret the two terms in the expression above thermodynamically. We have

$$\log K_{\text{eq}} = \underbrace{-\frac{\Delta E_0}{RT}}_{\text{enthalpic}} + \underbrace{\sum_i v_i \log \frac{z^{\ominus}_{j,0}}{N_{\text{A}}}}_{\text{entropic}}. \tag{9.7}$$

The first term compares the ground state energies of the different species and weights them accordingly; the second term, being the logarithm of a partition function, accounts for *all accessible states*, some of which will be populated at higher temperatures. Note that as $T \to 0$, the enthalpic contribution dominates.

We continue to write the partition function for a standard state ideal gas. But what, precisely, do we mean by this? Table 9.1 summarizes the partition functions that we have developed and the standard state conditions / zero-point energies that we must include when computing equilibrium constants.

| energetic contribution | molar quantity | standard conditions | zero-point |
|---|---|---|---|
| translation | $V$ | $V^{\ominus}$ | $\approx 0$ |
| rotation | | | |
| vibration | | | $\frac{\hbar\omega}{2}$ |
| electronic | | | $\epsilon_0$ |

Table 9.1: Our collection of partition functions so far.

## 9.2 Entropy, more generally

Because we have access to the Helmholtz free energy, we can compute other thermodynamic potentials in terms of the partition function. For example, because

$$-\left(\frac{\partial A}{\partial T}\right)_{N,V} = S(T) = k_{\mathrm{B}} \log Z + k_{\mathrm{B}} T \left(\frac{\partial \log Z}{\partial T}\right)_{N,V} \tag{9.8}$$

and recall again that

> Given a function $f(\beta)$,
>
> $$\frac{\mathrm{d}f}{\mathrm{d}T} = \frac{\mathrm{d}f}{\mathrm{d}\beta}\frac{\mathrm{d}\beta}{\mathrm{d}T} = -\frac{1}{k_{\mathrm{B}}T^2}\frac{\mathrm{d}f}{\mathrm{d}\beta}. \tag{9.9}$$
>
> This relation allows us to easily convert between $\frac{\partial}{\partial T}$ and $\frac{\partial}{\partial \beta}$.

Now, let's note that

$$\begin{aligned}\left(\frac{\partial \log Z}{\partial T}\right)_{N,V} &= -\frac{1}{k_{\mathrm{B}}T^2}\left(\frac{\partial \log Z}{\partial \beta}\right)_{N,V} \\ &= \frac{1}{k_{\mathrm{B}}T^2}\frac{\sum_\nu E(\nu) e^{-\beta E(\nu)}}{Z}\end{aligned} \tag{9.10}$$

and so

$$\begin{aligned}k_{\mathrm{B}}T\left(\frac{\partial \log Z}{\partial T}\right)_{N,V} &= \frac{1}{T}\frac{\sum_\nu E(\nu) e^{-\beta E(\nu)}}{Z} \\ &= -k_{\mathrm{B}} Z^{-1} \sum_\nu e^{-\beta E(\nu)} \log e^{-\beta E(\nu)}.\end{aligned} \tag{9.11}$$

The first term in (9.8) we can also write in this style:

$$k_{\mathrm{B}} \log Z = \frac{1}{Z} \sum_{\nu} e^{-\beta E(\nu)} \log Z \tag{9.12}$$

where we have used the fact that $\log Z$ is just a constant and $\langle C \rangle = C$ for any constant $C$. Combining these two expressions

$$S(T) = -k_{\mathrm{B}} \sum_{\nu} \frac{e^{-\beta E(\nu)}}{Z} \log \frac{e^{-\beta E(\nu)}}{Z}. \tag{9.13}$$

This expression now recognize; because $p(\nu) = Z^{-1} e^{-\beta E(\nu)}$, we have recovered the Shannon definition of the entropy,

$$S(T) = -k_{\mathrm{B}} \sum_{\nu} p(\nu) \log p(\nu). \tag{9.14}$$

This is the "information theoretic" definition of the entropy and it helps us understand the connect between entropy and *disorder*. Loosely,

More accessible configurations $\implies$ Higher entropy

In the next homework, you will prove that the entropy is maximal for uniform probability distributions. That fact should align well with your intuition about the relationship between entropy and temperature.

## 9.3 Obtaining other ensembles

We have used Legendre transforms to move between thermodynamic potentials. However, the Laplace transform also gave us a statistical way of moving between ensembles. Recall that the Laplace transform of $Z$ related the canonical and microcanonical partition functions, and we showed

$$Z(\beta) = \int \Omega(E) e^{-\beta E} dE \equiv \tilde{\Omega}(\beta). \tag{9.15}$$

The Laplace transform *integrates out* the energy from the expression for the microcanonical ensemble and produces a function that depends on $\beta$. These relations can be summarized in the diagram below.

$$\begin{tikzcd}
S(E,V,N) \arrow[r, "\text{Legendre}"] \arrow[d, "\text{ensemble}"] & A(T,V,N) \arrow[d, "\text{ensemble}"] \\
\Omega = e^{S(E,V,N)/k_{\mathrm{B}}} \arrow[r, "\text{Laplace}"] & \tilde{\Omega} = Z = e^{-\beta A(T,V,N)}
\end{tikzcd}$$

Of course, we want to explore other thermodynamic conditions. Consider an ensemble in which the energy and the number of particles both fluctuate. In this ensemble, we can do work to change particle number; that is, the first law says

$$dE = TdS - pdV + \mu dN. \tag{9.16}$$

Using the approach outlined above, let's postulate that the statistical weight for a configuration in this ensemble will be related to a Legendre transform of the entropy. That is,

$$p(\nu) \propto \exp\left[-k_{\mathrm{B}}^{-1}\left(\left(\frac{\partial S}{\partial E}\right)_{N,V} E(\nu) + \left(\frac{\partial S}{\partial N}\right)_{T,V} N(\nu)\right)\right]. \tag{9.17}$$

Each of these derivatives, of course, has a thermodynamic identity which can be read off from the corresponding total differentials. We conclude that, when the number of particles and the energy are both exchanging with a reservoir,

$$p(\nu) \propto \exp\left[-\beta E(\nu) + \beta\mu N(\nu)\right]. \tag{9.18}$$

A more formal derivation of this equation is possible by Taylor expanding the microcanonical partition function about $E_{\text{tot}}$ and $N_{\text{tot}}$.

## 9.4 Grand Canonical Ensemble

We have unraveled a new cumulant generating function. The *grand canonical partition function* is

$$\boxed{\Xi(T, V, \mu) = \sum_{\nu} e^{-\beta E(\nu) + \beta\mu N(\nu)}.} \tag{9.19}$$

As we might deduce, $\log \Xi$ contains statistical information about energy and number fluctuations. For example,

$$\left(\frac{\partial \log \Xi}{\partial \beta\mu}\right)_{\beta,V} = \frac{\sum_{\nu} N(\nu) e^{-\beta E(\nu)+\beta\mu N(\nu)}}{\sum_{\nu} e^{-\beta E(\nu)+\beta\mu N(\nu)}} = \langle N \rangle . \tag{9.20}$$

Similarly,

$$\left(\frac{\partial^2 \log \Xi}{\partial (\beta\mu)^2}\right)_{\beta,V} = \langle N^2 \rangle - \langle N \rangle^2 . \tag{9.21}$$

Just as we were able to relate the canonical partition function to a thermodynamic potential, we can use a Laplace transform to identify the grand canonical partition function's thermodynamic meaning. We simply write the partition function as a sum over $N$, as follows,

$$\begin{aligned} \Xi &= \sum_{N} e^{\beta\mu N} \sum_{N(\nu)=N} e^{-\beta E(\nu)}, \\ &= \sum_{N} e^{\beta\mu N - \beta A(N,V,T)}. \end{aligned} \tag{9.22}$$

The quantity inside the exponent is proportional to

$$\mu N - E + TS, \tag{9.23}$$

so using[*]

$$E = TS - pV + \mu N \tag{9.24}$$

---

[*] Why is this legitimate? Euler's theorem for homogeneous functions.

we conclude that only term that survives is $pV$. In other words,

$$\boxed{\Xi = e^{\beta pV} \implies \beta^{-1}\log\Xi = pV \qquad (9.25)}$$

# Lecture 10 Mathematical framework of thermodynamics

## Recap

- We derived the *grand canonical partition function*,

$$\Xi = \sum_{\nu} e^{-\beta E(\nu)+\beta\mu N(\nu)}. \tag{10.1}$$

- We showed that the logarithm of this function is a cumulant generating function for particle number statistics, that is,

$$\left(\frac{\partial \log \Xi}{\partial \beta\mu}\right)_{\beta,V} = \langle N \rangle\,; \quad \left(\frac{\partial^2 \log \Xi}{\partial \beta\mu^2}\right)_{\beta,V} = \langle N^2 \rangle - \langle N \rangle^2\,. \tag{10.2}$$

- We showed that the thermodynamic identity of $\Xi$ was related to the "grand potential",

$$\Xi = e^{\beta pV}. \tag{10.3}$$

## 10.1 Homogeneous functions and Gibbs-Duhem equation

The energy is extensive and it is a natural function of the quantities $(S, V, N)$. Recall that extensiveness means

$$E(\alpha S, \alpha V, \alpha N) = \alpha E(S, V, N), \tag{10.4}$$

mathematically, we say that $E$ is a homogeneous function of degree one. Euler's theorem for homogeneous functions says that for any such function,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)_{x_{j\neq i}} x_i. \tag{10.5}$$

Using the thermodynamic identity of the derivatives of $E$, we can conclude that

$$E = TS - pV + \mu N. \tag{10.6}$$

Furthermore, from the first law, we know that

$$dE = TdS - pdV + \mu dN, \tag{10.7}$$

therefore, from the expression above we must have that

$$SdT - Vdp + Nd\mu = 0, \tag{10.8}$$

or, dividing through by $N$, and defining $S/N \equiv s$ and $V/N \equiv v$,

$$sdT - vdp + d\mu = 0. \tag{10.9}$$

This is known as the Gibbs-Duhem equation; the relation places a constraint on phase equilibria and also tells us how to compute some new thermodynamic derivatives.

## 10.2 Isothermal-isobaric ensemble

Fixed temperature and fixed pressure constitutes one of the most common settings for chemistry. Let us once again use the same trick to postulate an expression for the statistical weight of a microstate at fixed temperature and pressure

$$p(\nu) \propto \exp\left[-k_{\mathrm{B}}^{-1}\left(\left(\frac{\partial S}{\partial E}\right)_{N,V} E(\nu) + \left(\frac{\partial S}{\partial V}\right)_{N,E} V(\nu)\right)\right]. \tag{10.10}$$

We know that since

$$dS = \frac{1}{T}dE + \frac{p}{T}dV - \frac{\mu}{T}dN \tag{10.11}$$

that we can write the corresponding partition function as

$$\boxed{\Delta(N,p,T) = \sum_{\nu} e^{-\beta E(\nu) - \beta p V(\nu)}.} \tag{10.12}$$

Let us carry out the program as before. This is a cumulant generating function for volume fluctuations,

$$\left(\frac{\partial \log \Delta}{\partial(-\beta p)}\right)_{\beta,N} = \frac{\sum_{\nu} V(\nu) e^{-\beta E(\nu) - \beta p V(\nu)}}{\sum_{\nu} e^{-\beta E(\nu) - \beta p V(\nu)}} = \langle V \rangle . \tag{10.13}$$

Similarly,

$$\left(\frac{\partial^2 \log \Delta}{\partial(-\beta p)^2}\right)_{\beta,N} = \langle V^2 \rangle - \langle V \rangle^2 . \tag{10.14}$$

Noticing that the variance is equivalent to

$$\langle V^2 \rangle - \langle V \rangle^2 = -\left(\frac{\partial \langle V \rangle}{\partial \beta p}\right)_{\beta,N}, \tag{10.15}$$

we obtain another fluctuation-response relation. In particular, the isothermal compressibility is related to volume fluctuations!

And once again, we can Laplace transform to understand the thermodynamic identity.

$$\Delta(N,p,T) = \sum_{V} e^{-\beta p V} \sum_{V(\nu)=V} e^{-\beta E} \tag{10.16}$$

which indicates that the quantity of interest will be proportional to

$$pV + A = pV + E - TS. \tag{10.17}$$

The resulting quantity is a natural function of $(p, T, N)$ and it will come as no surprise that it is the Gibbs free energy $G$.

$$\boxed{\Delta = e^{-\beta G} \implies -\beta^{-1}\log\Delta = G} \tag{10.18}$$

We now have a statistical mechanical expression for the Gibbs free energy.

## 10.3 Laplace, Laplace

The general definition of the Laplace transform of a function $f$ is

$$\tilde{f}(s) = \int_0^\infty f(t)e^{-st}dt. \tag{10.19}$$

The Laplace transform comes up in many measurements and the difficulty of inverting a Laplace transformtion is an outstanding problem for the interpretation of many experiments, and appears in, for example, two dimensional fluorescence correlation spectroscopy.

We have used it to convert the dependence of one partition function on a natural variable to a distinct variable. For example,

$$\tilde{\Omega}(\beta) = \int \Omega(E)e^{-\beta E}dE = Z(\beta). \tag{10.20}$$

In the thermodynamic limit, the minimal value of the exponent dominates and hence,

$$\tilde{\Omega}(\beta) = \int e^{-\beta E + \beta TS}dE \approx e^{-\beta(E_* - TS_*)} \tag{10.21}$$

will be dominated by the most likely energy $E_*$ when $N$ is large, we can deduce that

$$-\beta^{-1}\log Z(\beta) = A(N, V, T). \tag{10.22}$$

Approximating an integral of $e^{-Nf(x)}$ by the integrand at maximum value[*] $x_*$ is known as a *Laplace's method*; confusingly, it has nothing to do with the transform we introduced. We can do the same calculation for other ensembles. The Laplace transform of the canonical partition function, integrating out the number $N$, gives

$$\tilde{Z}(-\beta\mu) = \int Z(N)e^{\beta\mu N}dN = \Xi, \tag{10.23}$$

the *grand canonical partition function*, and

$$\Xi = \int e^{-\beta A + \beta\mu N}dN \approx e^{-\beta(A + \mu N)} = e^{\beta pV}. \tag{10.24}$$

Similarly, the Laplace transform of the canonical partition function, integrating out the volume $V$, gives

$$\tilde{Z}(\beta p) = \int Z(V)e^{-\beta pV}dV = \Delta, \tag{10.25}$$

---

[*] or the minimum value of the exponent…

the *isothermal-isobaric* partition function, and

$$\Delta = \int e^{-\beta A - \beta p V} dV \approx e^{-\beta(A+pV)} = e^{-\beta G}. \tag{10.26}$$

These transforms give us a universal way of moving between thermodynamic potentials. Essentially all thermodynamic relations can be derived from the content in the following diagram.

$$\begin{tikzcd}
S(E,V,N) \arrow[r, "-\left(\frac{\partial S}{\partial E}\right)_{V,N}E"] \arrow[d, "-\left(\frac{\partial S}{\partial E}\right)_{V,N}E-\left(\frac{\partial S}{\partial N}\right)_{E,V}N"] & A(T,V,N) \arrow[d, "-\left(\frac{\partial A}{\partial V}\right)_{T,N}V"] \\
\Phi(T,V,\mu) \arrow[r, "-\left(\frac{\partial \Phi}{\partial V}\right)_{T,\mu}V-\left(\frac{\partial \Phi}{\partial \mu}\right)_{T,V}\mu"] & G(T,p,N)
\end{tikzcd}$$

Similarly, we can consider the ensembles:

$$\begin{tikzcd}
\Omega = e^{S/kb} \arrow[r, "\int \cdot e^{-\beta E} dE"] & Z = e^{-\beta A} \arrow[dl, "\int \cdot e^{\beta \mu N} dN"] \arrow[d, "\int \cdot e^{-\beta p V} dV"] \\
\Xi = e^{\beta p V} & \Delta = e^{-\beta G}
\end{tikzcd}$$

# Lecture 11 Quantum statistics

## Recap

Last time, we introduced the isobaric-isothermal ensemble and discussed Laplace transforms (which allow us to move between statistical ensembles) and Legendre transforms (which allow us to move between thermodynamic potentials).

- In the isobaric-isothermal ensemble, the probability of a microstate $\nu$ is given by

  $$p(\nu) = \frac{e^{-\beta E(\nu) - \beta p V(\nu)}}{\Delta} \tag{11.1}$$

  where

  $$\Delta = \sum_{\nu} e^{-\beta E(\nu) - \beta p V(\nu)} \tag{11.2}$$

  is the associated partition function.

- The logarithm of $\Delta$ is a cumulant generating function for the volume. That is

  $$\left( \frac{\partial \log \Delta}{\partial(-\beta p)} \right)_{\beta, N} = \langle V \rangle \tag{11.3}$$

  and higher derivatives yield higher cumulants.

## 11.1 Realistic quantum particles

Let us replay the narrative of the course thus far. We asserted that, to the best of knowledge, there is a fundamentally accurate model of all molecular physics, the Schrödinger equation. One equation to rule them all, if you will:

$$E\Psi(r) = V(r)\Psi(r) - \frac{\hbar^2}{2m}\nabla^2\Psi(r). \tag{11.4}$$

This equation, of course, is a partial differential equation in a high-dimensional space—it is all but impossible to solve for complex systems, numerically or analytically. Using simple models of the energetics, we obtained expressions for the molecular partition function that have enabled us to compute thermodynamic properties of non-interacting systems, from ideal gas properties to equilibrium constants.

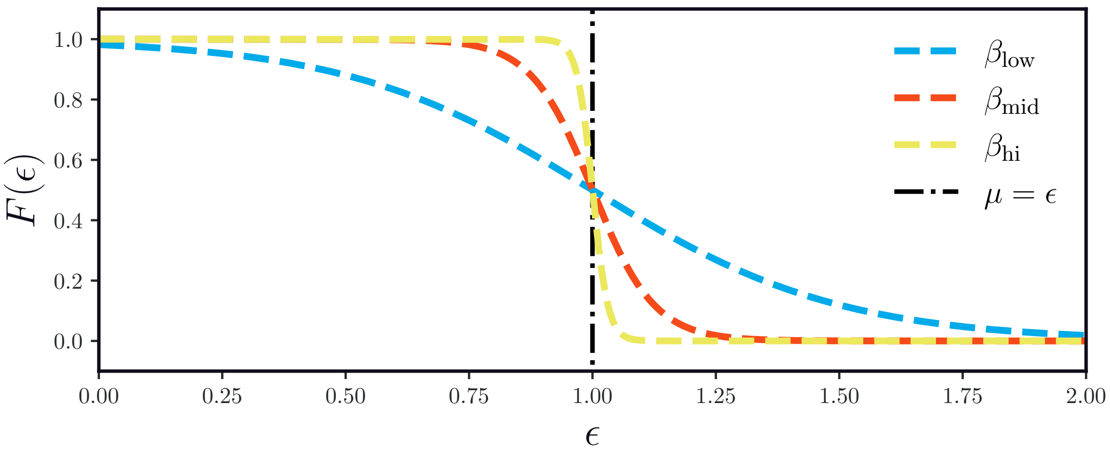


Figure 11.1: Fermi-Dirac distribution

We have done all of this work without paying much attention to realistic quantum particles. Rather than energy levels and occupancies, we should instead be solving single-particle Schrödinger equations, arriving at functions,

$$h(r)\psi_j(r) = \epsilon_j \psi_j(r). \tag{11.5}$$

So if we consider a two particle quantum mechanical system with wavefunction

$$\Psi(r_1, r_2), \tag{11.6}$$

what is the probability of the state specified by $\Psi$?

In quantum mechanics, this is a simple matter—the probability is determined by the amplitude of the wavefunction

$$\begin{aligned} \text{Prob.}(1,2) &= |\Psi(r_1, r_2)|^2, \\ &= |\Psi(r_2, r_1)|^2, \\ &= \text{Prob.}(2,1), \end{aligned} \tag{11.7}$$

by symmetry. So, the only thing that can change at the level of the wave function when swapping $r_1$ and $r_2$ is the phase

$$\Psi(r_1, r_2) = e^{i\phi}\Psi(r_1, r_2). \tag{11.8}$$

For electrons, protons, and neutrons, $\phi = \pi$, meaning

$$\Psi(r_1, r_2) = -\Psi(r_1, r_2). \tag{Fermions}$$

For photons and bound pairs of Fermions, $\phi = 0$, meaning

$$\Psi(r_1, r_2) = \Psi(r_1, r_2). \tag{Bosons}$$

If we write the wavefunction in terms of single-particle eigenfunctions, we must symmetrize appropriately:

$$\Psi_{\text{fermion}}(r_1, r_2) = \frac{1}{\sqrt{2}}\left(\psi_j(r_1)\psi_k(r_2) - \psi_k(r_1)\psi_j(r_2)\right), \tag{11.9}$$

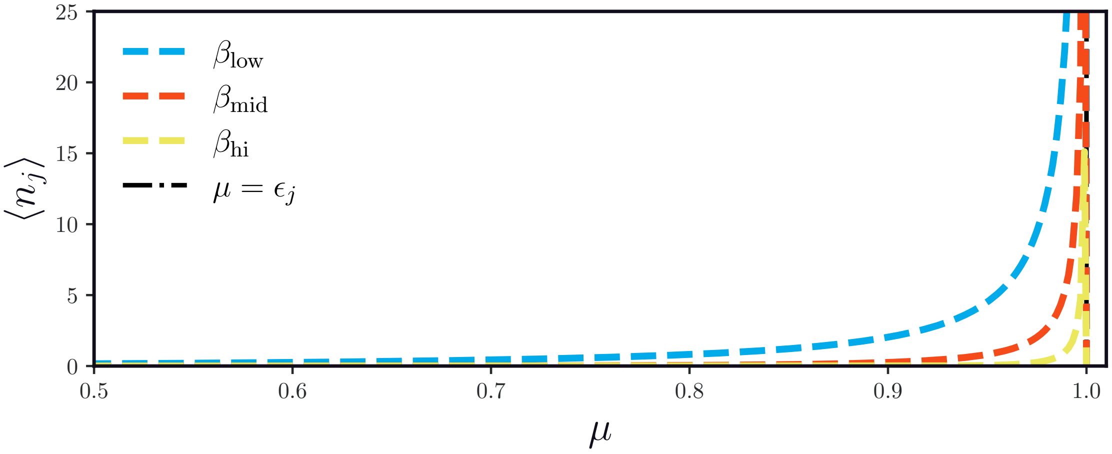


Figure 11.2: Bose-Einstein distribution.

and

$$\Psi_{\text{boson}}(r_1, r_2) = \frac{1}{\sqrt{2}} \left( \psi_j(r_1)\psi_k(r_2) + \psi_k(r_1)\psi_j(r_2) \right). \tag{11.10}$$

This has the well-known physical consequence of Pauli exclusion. If $j = k$, then the fermionic wavefunction is the zero function, meaning that there can only be one particle occupying that energy eigenstate.

We know how to compute occupancy statistics using the grand canonical partition function,

$$\begin{aligned} \Xi_{\text{fermion}} &= \sum_{n_0=0}^{1} \sum_{n_1=0}^{1} \cdots \sum_{n_{N_l}=0}^{1} e^{-\beta \sum_j n_j \epsilon_j + \beta\mu \sum_j n_j} \\ &= \prod_{j=0}^{N_l} 1 + e^{\beta(\mu - \epsilon_j)}. \end{aligned} \tag{11.11}$$

In this expression, $N_l$ denotes the number of energy levels. Computing the average occupancy of energy level $j$ is easy—we simply use

$$\left( \frac{\partial \log \Xi}{\partial \beta(\mu - \epsilon_j)} \right) = \langle n_j \rangle = \frac{e^{\beta(\mu - \epsilon_j)}}{1 + e^{\beta(\mu - \epsilon_j)}}. \tag{11.12}$$

## 11.2 Surprising quantum statistics

Bosons, on the other hand, have totally unrestricted occupancy statistics. If the occupancy of the ground state energy $\epsilon_0$ has no restrictions, does it mean that a *macroscopic* number of particles could occupy the ground state simultaneously? If this is the case, then

$$\Omega(N, V, E) = 1 \implies S = 0. \tag{11.13}$$

Because we know how to compute occupancy statistics using the grand canonical partition function,

$$\begin{aligned}\Xi_{\text{boson}} &= \sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty} e^{-\beta\sum_j n_j\epsilon_j+\beta\mu\sum_j n_j} \\ &= \prod_{j=0}^{N}\frac{1}{1-e^{-\beta(\epsilon_j-\mu)}}.\end{aligned} \tag{11.14}$$

Once again, computing the average occupancy of energy level $j$ is easy—we simply use

$$\left(\frac{\partial \log \Xi}{\partial \beta(\mu-\epsilon_j)}\right) = \langle n_j\rangle = \frac{1}{e^{\beta(\epsilon_j-\mu)}-1}. \tag{11.15}$$

As $\mu \to \epsilon$, there is a divergence! A single energy level is occupied by a macroscopic number of bosonic particles.

Bose-Einstein condensates are a bizarre state of matter, only accessible at very low temperatures and pressures. In the lab, they have been realized by cooling bosons like $^4$He to low temperatures ($< 2.17K$) or vapors of $^{87}$Rb to nanokelvin temperatures. Superfluidity is one of the consequences of a macroscopic state in which every particle is identical.

One of the most amazing experiments exploiting the strange properties of BECs was conducted in the research group of Prof. Lene Hau at Harvard. In this set of experiments, they used a sodium vapor BEC to alter the refractive index of the material to dramatically slow the propagation of a light pulse. Prof. Hau wrote a very readable account of the approach for Scientific American.

# Lecture 12 Impact of quantum statistics on solids at low temperature

## Recap

1. Single particle Hamiltonian with eigenstates

$$h(r)\psi_j = \epsilon_j \psi_j \tag{12.1}$$

for a non-interacting, indistinguishable system

$$\mathcal{H} = h(r_1) + \cdots + h(r_N) \tag{12.2}$$

and the total energy can be written in terms of the occupation variables $n_j$, which quantifies the number of particles in energy eigenstate $\psi_j$,

$$E = \sum_j \epsilon_j n_j. \tag{12.3}$$

## 12.1 Bosons and Fermions

Let's first repeat the calculation that we did last time. We computed the grand canonical partition function for bosons,

$$\begin{aligned} \Xi_{\text{boson}} &= \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} e^{-\beta \sum_j n_j \epsilon_j + \beta\mu \sum_j n_j} \\ \Xi_{\text{boson}} &= \sum_{n_0=0}^{\infty} e^{-\beta n_0 \epsilon_0 + \beta\mu n_0} \sum_{n_1=0}^{\infty} e^{-\beta n_1 \epsilon_1 + \beta\mu n_1} \cdots \sum_{n_N=0}^{\infty} e^{-\beta n_N \epsilon_N + \beta\mu n_N}, \\ &= \prod_{j=0}^{N} \frac{1}{1 - e^{\beta(\mu - \epsilon_j)}}. \end{aligned} \tag{12.4}$$

Computing the average occupancy of energy level $j$ is easy—we simply use

$$\left( \frac{\partial \log \Xi}{\partial \beta(\mu - \epsilon_j)} \right) = \langle n_j \rangle = \frac{1}{e^{\beta(\epsilon_j - \mu)} - 1}. \tag{12.5}$$

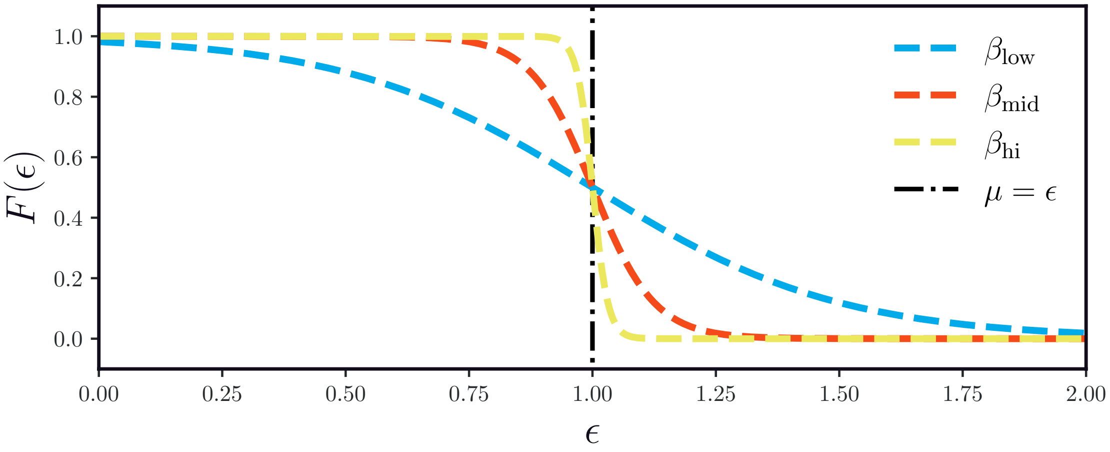


Figure 12.1: Fermi-Dirac distribution

We noted that there is a constraint that emergies from the symmetery of the wave-function that indicates that fermions can only singly occupy an energy eigenstate. This led us to the expression

$$\begin{aligned}\Xi_{\text{fermion}} &= \sum_{n_0=0}^{1}\sum_{n_1=0}^{1}\cdots\sum_{n_N=0}^{1} e^{-\beta\sum_j n_j\epsilon_j+\beta\mu\sum_j n_j},\\ \Xi_{\text{fermion}} &= \sum_{n_0=0}^{1} e^{-\beta n_0\epsilon_0+\beta\mu n_0}\sum_{n_1=0}^{1} e^{-\beta n_1\epsilon_1+\beta\mu n_1}\cdots\sum_{n_N=0}^{1} e^{-\beta n_N\epsilon_N+\beta\mu n_N},\\ &= \prod_{j=0}^{N} 1+e^{-\beta(\epsilon_j-\mu)}.\end{aligned} \tag{12.6}$$

Both of these calculations are considerably simplified because we can *factorize*. Secondly, we used the fact that derivatives of log-partition functions give statistical information. So, we observe that

$$\left(\frac{\partial \log \Xi_{\text{fermion}}}{\partial \beta(\mu-\epsilon_j)}\right) = \langle n_j\rangle = \frac{e^{\beta(\mu-\epsilon_j)}}{1+e^{\beta(\mu-\epsilon_j)}}. \tag{12.7}$$

We obtained an expression for the occupancy of $\epsilon_j$ that depends on the inverse temperature $\beta$ and the chemical potential $\mu$. Let us examine the function for the occupancy at energy level $\epsilon$, now as a function of energy, shown in Fig. 12.1. The limiting behavior here is quite clear: as $T \to 0$, the Fermi function becomes a step function,

$$F_0(\epsilon) = \begin{cases} 1 & \epsilon < \mu_0, \\ 0 & \epsilon > \mu_0. \end{cases} \tag{12.8}$$

When the temperature is low ($\beta \to \infty$) and the chemical potential $\mu$ is such that the ground state

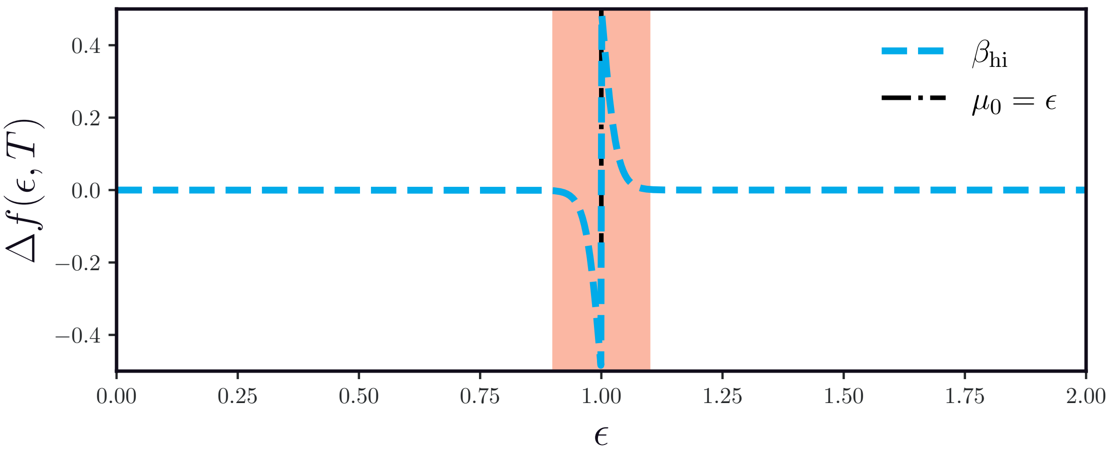


Figure 12.2: A plot showing $\Delta f$ as a function of $\epsilon$ at low temperature.

is populated $\mu \approx \mu_0$, there will be fluctuations in the energy on the scale of $k_{\rm B}T$. We can write

$$\epsilon \approx \epsilon_0 + k_{\rm B}T. \tag{12.9}$$

First, let us split the Fermi function into a temperature indepedent and temperature dependent part:

$$F(\epsilon) = F_0(\epsilon) + \Delta f(\epsilon, T), \tag{12.10}$$

plotted in Fig. 12.2. The low temperature fluctuations in energy (and hence the heat capacity) will be determined by the region near the transition.

What kinds of energies do we need to start populating higher energy levels? One way of rationalizing this question is to consider the energies required for an electron to occupy states with $\epsilon = \mu_0$, which will require temperatures of $T \approx \mu_0/k_{\rm B}$. For solid copper, this would correspond to a temperature of about 80000K.

This fact suggests that at low temperature, very few electrons will have energies on the order of $\mu_0$ so we can assume that they are *independent*. Of course, being quantum particles, they are also indistinguishable by their very nature. In other words, it is reasonable to treat metals as an ideal gas of fermions. Of course, this has implications for physically measurable quantities like the heat capacity.

## 12.2 Heat capacity for low temperature solids

So, let us calculate the heat capacity at low temperature assuming this ideal gas of fermions model. Luckily, we now know how to deal with this situation. First, the single particle states are roughly those of the particle in a box model,

$$\epsilon = \frac{h^2}{8mL^2}\left(m_x^2 + m_y^2 + m_z^2\right), \tag{12.11}$$

where the quantum numbers $m_\alpha$ are positive integers. The eigenstates of this model are standing waves and so it is more convenient to work with wavevectors,

$$\vec{k} = \frac{\pi}{L}\left(m_x\hat{x} + m_y\hat{y} + m_z\hat{z}\right). \tag{12.12}$$

Using the orthogonality of the unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$, we obtain a simple expression for the energy

$$\epsilon = \frac{\hbar^2}{2m}k^2 \tag{12.13}$$

where $k = |\vec{k}|$. It is a calculation you have likely seen before (and one we will do again in the next few lectures), but we can count the number of wavevectors with norm $k$. This is the "density of states", which we will denote $g$,

$$g(k) = \frac{V}{2\pi^2}k^2. \tag{12.14}$$

Once you have access to the density of states as a function of $k$, you can (and will on a future homework!) derive the density of states as a function of the energy $\epsilon$. For now, we will just call the expression $g(\epsilon)$ without actually using the formula.

Of course, if we have $g(\epsilon)$, we can now write an expression for the average energy

$$\langle E\rangle \propto \int_0^\infty \epsilon g(\epsilon)F(\epsilon)d\epsilon. \tag{12.15}$$

Next time, we will focus on extracting the temperature dependence from this expression when $T$ is small.

# Lecture 13 Classical limit

## Recap

1. Wrote expressions for

$$\begin{aligned} \langle n_j \rangle_{\text{fermions}} &= \frac{1}{e^{\beta(\epsilon_j - \mu)} + 1} \\ \langle n_j \rangle_{\text{bosons}} &= \frac{1}{e^{\beta(\epsilon_j - \mu)} - 1} \end{aligned} \tag{13.1}$$

## 13.1 Low temperature heat capacity

At low temperatures, we have

$$\langle E \rangle = \int_0^\infty \epsilon g(\epsilon) \left[ F_0(\epsilon) + \Delta f(\epsilon, T) \right] d\epsilon \tag{13.2}$$

As we can see, $\Delta f$ is non-zero only when $\epsilon \approx \mu_0$. What do we do in such a situation? This is a wonderful opportunity to Taylor expand the density of states around $\mu_0$. Looking only at the term that varies,

$$\langle E \rangle \propto \text{Const.} + \int_0^\infty \epsilon \big( g(\mu_0) + g'(\mu_0)(\epsilon - \mu_0) + \dots \big) \Delta f(\epsilon, T) d\epsilon. \tag{13.3}$$

We are interested in the $T$ dependence when calculating a heat capacity, so setting $x = \beta(\epsilon - \mu_0)$, we obtain

$$\langle E \rangle \propto \text{Const.} + \int_0^\infty (\mu_0 + k_B T x) \big( g(\mu_0) + g'(\mu_0) k_B T x + \dots \big) \Delta f(\mu_0 + k_B T x, T) k_B T dx. \tag{13.4}$$

But now, think graphically about the integral: we are multiplying by $\Delta f$ which is an odd function. That means that anything with an even power of $x$ will vanish in the integral. If we focus on the scaling with $T$, we see that

$$\langle E \rangle \propto \text{Const.} + AT^2 + BT^4. \tag{13.5}$$

This suggests that

$$C_v(T) = \frac{\partial \langle E \rangle}{\partial T} \propto T \tag{13.6}$$

when $T$ is very small. This is consistent with experimental observations; there are three distinct regimes summarized in the Fig. 14.6.

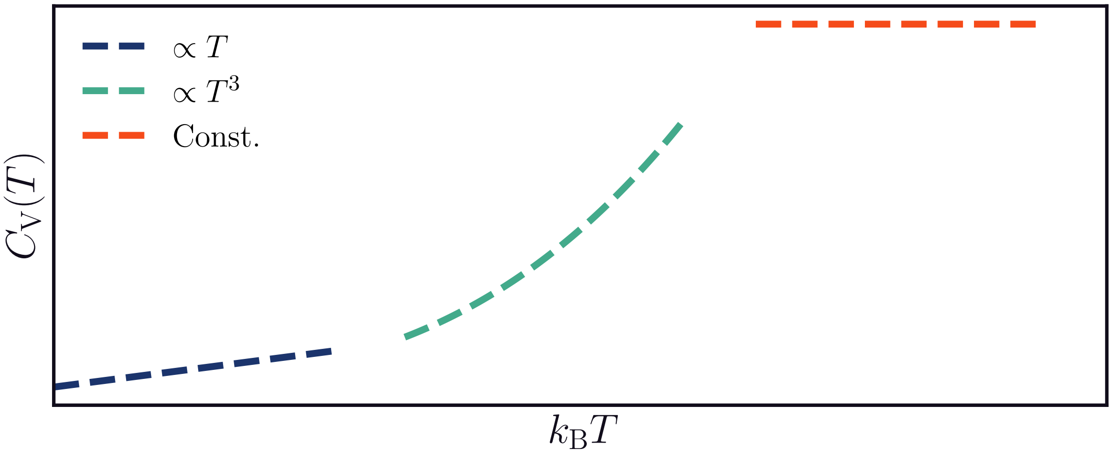


Figure 13.1: Three regimes of heat capacity scaling with $T$.

## 13.2 When is the boson-fermion distinction important?

Why don't we always care about the distinction between bosons and fermions? In the *classical limit*, the constraints on occupancy cease to be important because the probability of placing more than one particle in a given state is very small. That is, we require

$$\langle n_j \rangle_{\text{bosons}} = \frac{1}{e^{\beta(\epsilon_j - \mu)} - 1} \ll 1, \tag{13.7}$$

which means

$$\langle n_j \rangle_{\text{bosons}} \approx \frac{1}{e^{\beta(\epsilon_j - \mu)}}. \tag{13.8}$$

Similarly, for fermions in this regime,

$$\langle n_j \rangle_{\text{fermions}} = \frac{1}{e^{\beta(\epsilon_j - \mu)} + 1} \approx \langle n_j \rangle_{\text{bosons}}. \tag{13.9}$$

We conclude that this distinction is only important when a small number of accessible states are available at a given temperature.

## 13.3 Classical limit

The occupancy statistics matter *only* when there are correlations among quantum particles that result from simultaneous occupancy of a single state. When $\langle n_j \rangle \approx 0$, meaning that any one state is unlikely to be occupied, these correlations will not manifest. Which conditions lead to this behavior?

We will first investigate this for an ideal gas. Let's write the particle in a box energy function in a plane wave expansion: Consider a box with side length $L$ in each of its three dimensions. We first introduce an expansion into a basis of plane waves,

$$e^{i\vec{k}\cdot\vec{r}} \qquad \vec{k} = \frac{\pi}{L}\left(n_x\vec{x} + n_y\vec{y} + n_z\vec{z}\right). \tag{13.10}$$

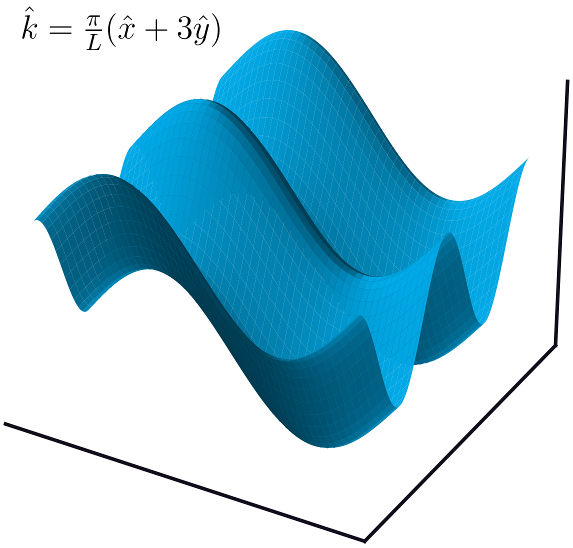


Figure 13.2: An example plane wave.

This vector indexes normal modes with $n_d$ nodes in dimension $d$; we know that this is the form of solutions to the particle in the box Schrödinger equation, so it's a natural basis for the problem. The magnitude of this plane-wave determines the energy of the corresponding particle in the box state because we can write

$$\epsilon(k) = \frac{\hbar^2}{2m} k^2 \tag{13.11}$$

where

$$|\vec{k}| \equiv k = 2\pi/\lambda. \tag{13.12}$$

In the last equality, we are writing the wavelength in terms of wavenumbers where $\lambda$ is simply the spatial frequency of the wave.

The energy of a state is determined entirely by $k$. However, there are multiple sets of quantum numbers that correspond to a single energy (see Fig. 13.3). In order to account for this degeneracy, we need to compute the density of states,

$$g(k)dk = \text{number of } \vec{k} \text{ with } |\vec{k}| \in [k, k + dk]. \tag{13.13}$$

Because $g$ is a denisty, if we want to get a number we must multiply by a volume element $dk$. For a macroscopic system, $\pi/L$ is small, meaning that we can approximate the spacing between wavevectors as continuous. In order to compute the density of states we need to compute,

$$\begin{aligned} g(k) &= (\text{density of modes}) \times (\text{surface area of octant}) \\ &= \left(\frac{L}{\pi}\right)^3 \times \frac{1}{8} 4\pi k^2. \end{aligned} \tag{13.14}$$

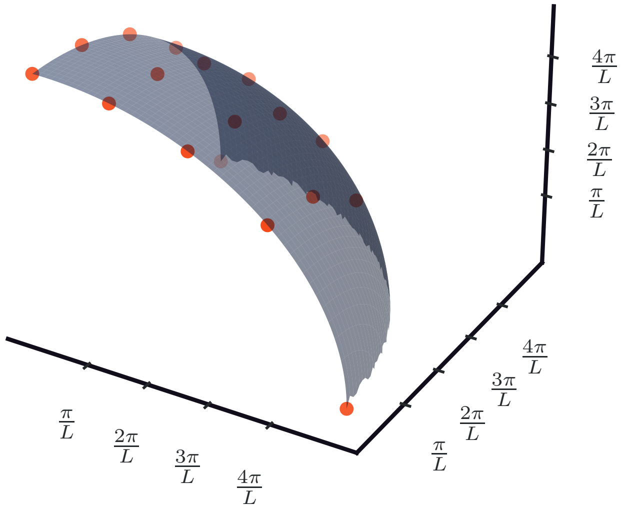


Figure 13.3: All wavevectors with the same energy form an octant.

We conclude that

$$g(k)dk = \frac{V}{2\pi^2}k^2, \tag{13.15}$$

so, the density of states is a quadratic function of the wavenumber.

Of course, once we have this expression, we can compute things. For example,

$$\begin{aligned} \langle N \rangle &= \int_0^\infty g(k) e^{-\beta\epsilon(k)+\beta\mu} dk \\ &= \frac{V}{2\pi^2} e^{\beta\mu} \int_0^\infty k^2 e^{-\beta\hbar^2 k^2/2m} dk \\ &= e^{\beta\mu} V \left(\frac{2\pi m}{\beta h^2}\right)^{3/2}. \end{aligned} \tag{13.16}$$

To simplify this expression, we evaluated the Gaussian moment integral

$$\int_0^\infty x^2 e^{-\alpha x^2} = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}. \tag{13.17}$$

Furthermore, once we have the average particle number, we can compute the density. We obtain

$$\rho = \frac{\langle N \rangle}{V} = e^{\beta\mu}\left(\frac{h}{\sqrt{2\pi m k_{\mathrm{B}} T}}\right)^{-3}. \tag{13.18}$$

We now recognize the thermal de Broglie wavelength,

$$\lambda_{\mathrm{T}}^{3}\rho = e^{\beta\mu}. \tag{13.19}$$

So, when is $\langle n_j \rangle \ll 1$? When $e^{\beta\mu} \ll 1$, which means that

$$\rho_{\text{classical}} \ll \lambda_{\mathrm{T}}^{-3}. \tag{13.20}$$

Quantum statistics matter when there is more than one particle per (thermal wavelength)$^3$.

# Lecture 14 Phenomenology of phase transitions

## Recap

1. Derived the classical limit and showed that the boson / fermion distinction does not have an impact on occupancy statistics when

$$\rho_{\text{classical}} \ll \frac{1}{\lambda_{\text{T}}^3} \tag{14.1}$$

   which means that quantum statistics matter only when there is *on average* more than one particle per (thermal wavelength)$^3$.

2. Introduced the idea of phase transitions and discussed sharp transitions in free energies.

## 14.1 Sharp transitions

We saw that as the chemical potential was tuned in a fermionic system, the quantum correlations led to a sharp transitions between the occupied and unoccupied levels. It turns out that this is a rather generic phenomenon. The sharp transitions with which we are most familiar are phase transitions.

Consider the phase diagram for water in the $(\rho, T)$ plane Fig. 14.1. There is a "non-analytic" cross-over in the density as temperature is changed. How can this happen? Let us think about this phenomenon like statistical mechanicians, starting with an expression for the classical, canonical partition function:

$$Z_{\text{liq}} = \int_{\text{liq}} e^{-\beta U(\boldsymbol{r}^N)} d\boldsymbol{r}^N. \tag{14.2}$$

Here the notation

$$\boldsymbol{r}^N = (\vec{r}_1, \ldots \vec{r}_N) \tag{14.3}$$

denotes the three-dimensional coordinates of the system. We know that the probability of the liquid state can be computed as

$$\text{Prob.(liq)} = \frac{Z_{\text{liq}}}{Z}. \tag{14.4}$$

The probability of the liquid state as a function of temperature has a sharp transition as $\beta \to \beta_{\text{vapor}}$. But this seems contrary to the mathematical expression that we have written because the probability is a ratio of smooth functions. Non-analytic behavior can *only* emerge in the limit $N \to \infty$.

So, let us think more explicitly about correlations. The system has local correlations, but we can divide into statistically uncorrelated regions if those regions are sufficiently large. Then we can write

$$Z = q_{\text{cell}}^M \tag{14.5}$$

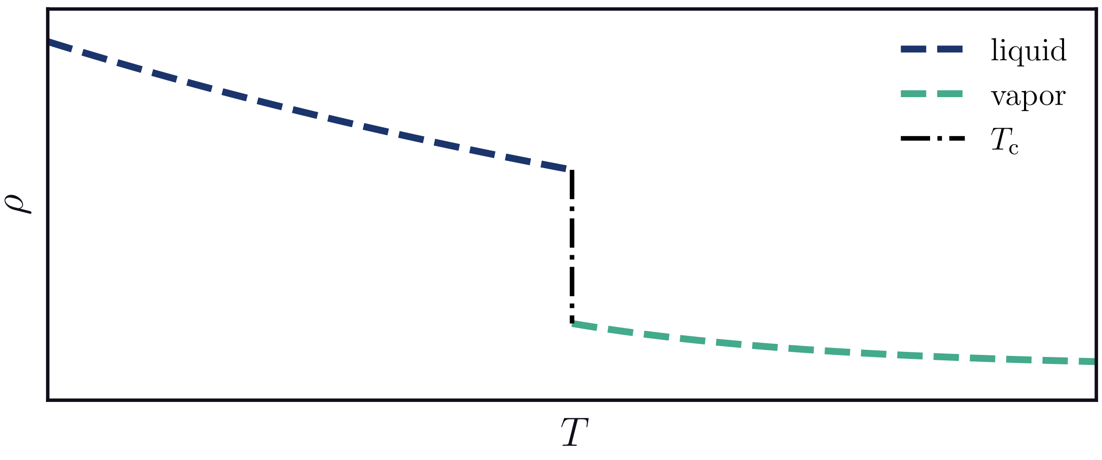


Figure 14.1: A schematic phase diagram for a liquid-vapor transition.

which of course means that we can write the Helmholtz free energy as

$$-\beta A = M \log q_{\text{cell}}(\beta). \tag{14.6}$$

The quantity $A/M$ is a smooth function of $\beta$, which would seem to suggest that there can be no phase transitions. Where did we go wrong? *Phase transitions involve correlations that are macroscopic in extent!*

This idea, that phase transitions involve divergent correlation lengths, is the foundation of the modern theory of phase transitions. We will explore this concept in much more depth in the next few lectures.

## 14.2 Phase transitions

Last time we discussed liquid-vapor transitions in order to frame the discussion of phase transitions. Magnets are another system often used to model phase transitions; surprisingly liquid-vapor and demagnetization transition are deeply related, as we will shortly see. Magnetic systems make the relevant symmetries apparent.

At low temperatures, magnets retain their magnetization. That is, if we were to apply a strong external field $h_z$ along the $z$-axis of the magnet so that, microscopically, the spins were mostly aligned with $z$, then the spin-up configuration would remain dominant. Now, imagine doing that experiment with the field oriented in the opposite direction, along the $-z$-direction. We would obtain a configuration that was predominately spin-down and it would remain so at sufficiently low temperatures. This picture is summarized in Fig. 14.2; the field couples to the spins of the system, which are shown as a microscopic detail in the macroscopic magnet. If this magnet initially has $\langle M \rangle > 0$, and we apply a field to reverse the magnet so that $\langle M \rangle < 0$, then, even when we turn off the external field, the magnetization remains reversed.

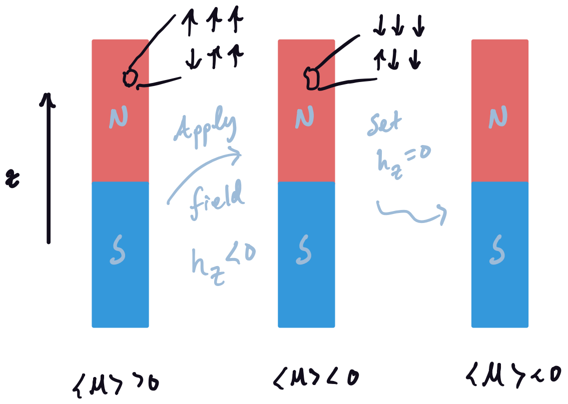


Figure 14.2: Minimal model of a magnet. A collection of spins are aligned along an axis. Upon heating, the spins adopt random orientations and demagnetize.

Is this situation consistent with the statistical mechanics perspective? Let us denote a given configuration of the magnet with a positive average spin $M$ by ↑. We should compute, at fixed $\beta$,

$$p(\uparrow) \propto e^{-\beta E(\uparrow)}, \tag{14.7}$$

and compare this quantity to

$$p(\downarrow) \propto e^{-\beta E(\downarrow)}, \tag{14.8}$$

where ↓ denotes the configuration ↑ where all the spins have been flipped. If $h_z = 0$, then the energies are equal by symmetry. That is,

$$E(\uparrow) = E(\downarrow) \text{ when } h_z = 0. \tag{14.9}$$

In this scenario we say, below the critical temperature, there is *spontaneous symmetry breaking*. In the limit $N \to \infty$, only one of the two macroscopic states, positive magnetization or negative magnetization, will be occupied. Despite the fact that the two configurations are equally probable, they will not interconvert. We can understand the origin of this phenomenon by looking at the free energy as a function of the magnetization, as plotted in Fig. 14.3

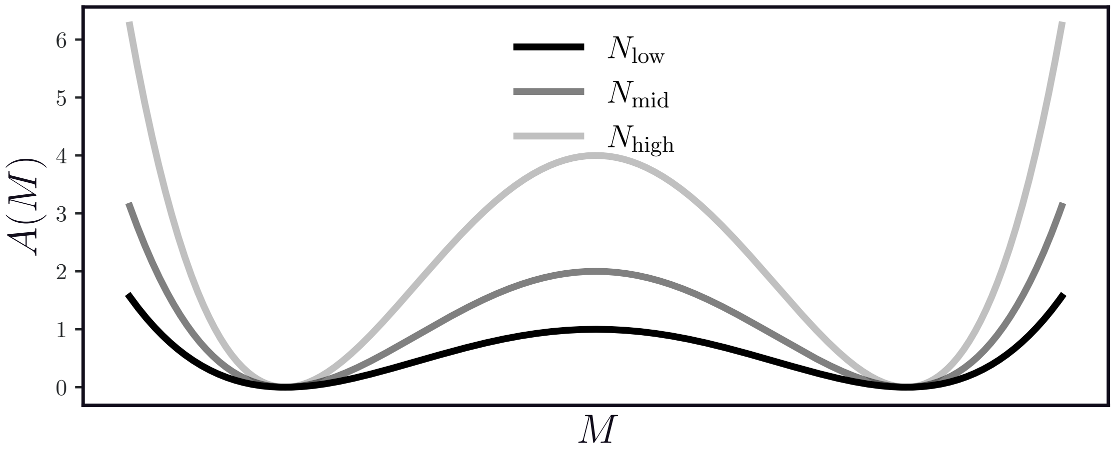


Figure 14.3: Schematic of the Helmholtz free energy as a function of magnetization for a simple model of a magnet *below* the critical temperature. There is a barrier that grows with the total number of spins.

Recall that we can relate the free energy to the probability of a particular macrostate. We know that

$$p(M) = e^{-\beta A(M)}. \tag{14.10}$$

If we examine the plot in Fig. 14.3, we see that in the limit $N \to \infty$, only the states with $|\langle M \rangle|/N \approx 1$ are populated.

To view this problem from a slightly different perspective, we look at the average magnetization at low temperature as a function of the external field $h_z$. A small external field breaks the symmetry as shown in Fig. 14.4. This quantity is closely related to $\chi$, the magnetic susceptibility, which measures how the average magnetization responds to an external field. As with other response-like quantities, the fact that the magnetic susceptibility diverges is indicative of divergent fluctuations. We will encounter this concept again.

## 14.3 Universal scaling at phase transitions

In fact, the connection runs quite deep. We can quantify how similar these curves are by looking at the scaling of magnetization with respect to temperature near the critical temperature. This is, we plot

$$|\langle M \rangle| \propto |T - T_{\text{c}}^{\text{mag}}|^{b_{\text{mag}}}, \tag{14.11}$$

which is shown in Fig. 15.1 We could also measure

$$|\rho - \rho_{\text{c}}| \propto |T - T_{\text{c}}^{\text{liq}}|^{b_{\text{liq}}}. \tag{14.12}$$

Experimentally, and quite remarkably,

$$b_{\text{liq}} = b_{\text{mag}} \equiv \text{"}\beta\text{"}. \tag{14.13}$$

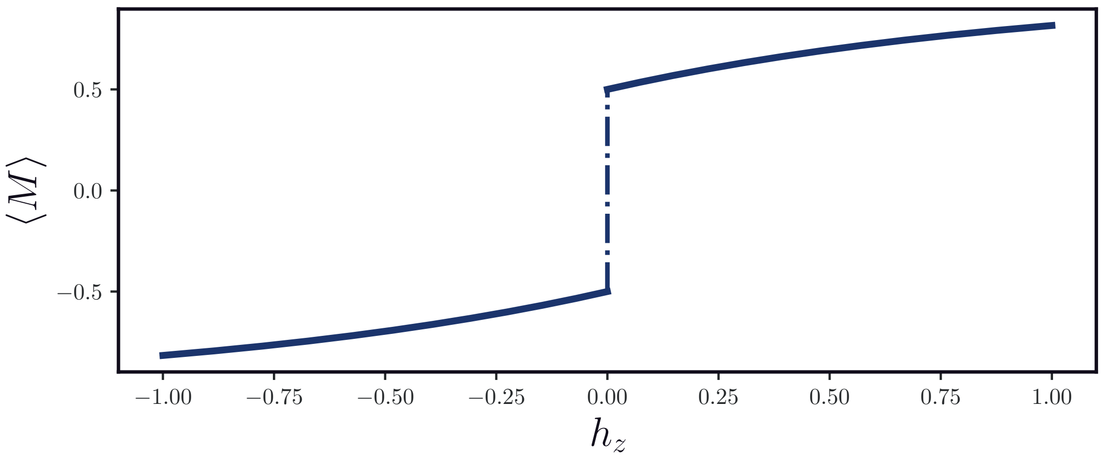


Figure 14.4: The average magnetization as a function of magnetization at low temperature.

I put the name of this exponent in quotes to emphasize that this ”$\beta$” is not the inverse temperature, but rather the name of a *critical exponent*. This unfortunate notation is, however, an entrenched historical practice. For a large variety of microscopically distinct materials,

$$\beta = 0.32. \tag{14.14}$$

There are sharp transitions in averages with respect to coupling parameters seen in the phase transitions we have discussed. This means that as one tunes $T$ close to the critical temperature $T_c$, there is a very large change in the magnetization (and in energy). Mathematically,

$$\left| \frac{\partial \langle M \rangle}{\partial T} \right| \gg 1 \text{ near } T_{\mathrm{c}}. \tag{14.15}$$

These large derivatives mean large changes in heat capacity. We show the characteristic divergence of the heat capacity in Fig. 14.6. This divergence follows a power law experimentally

$$C_{\mathrm{V}} \propto |T - T_{\mathrm{c}}|^{-\alpha}, \quad \alpha = 0.11, \tag{14.16}$$

again for a variety of microscopically distinct materials.

Phase transitions show *universality*. The scaling behavior near the critical point is independent of the microscopic details.

## 14.4 Ising model

The observation of universality motivates studying minimal models of phase transitions. For system of magnetic spins, we assume there are two distinct contributions to the overall energy: 1) a coupling to the external field hereon denoted $h$ and 2) local spin-spin coupling between nearby spins. Let us

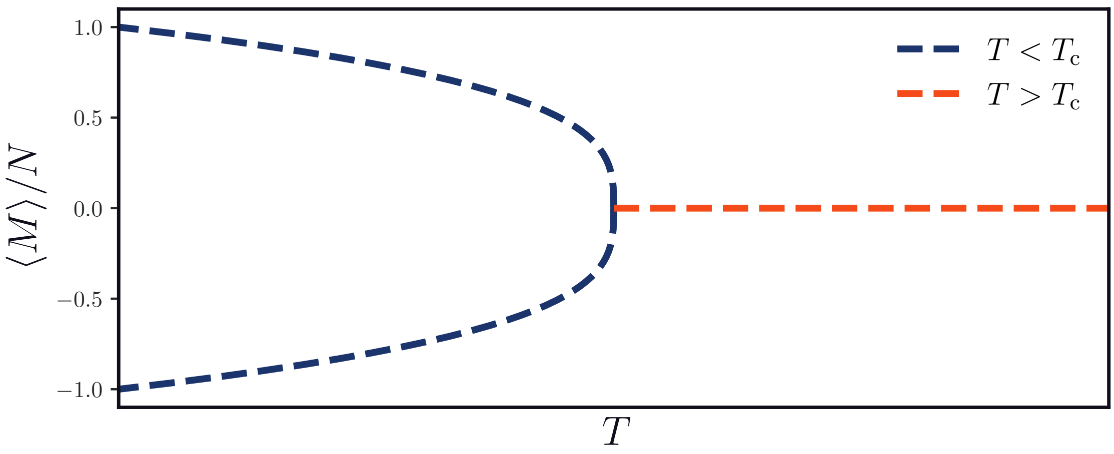


Figure 14.5: Scaling of the average magnetization near the critical temperature.

denote the spins, which are on a square lattice, by $\sigma_i$. Each spin can take a value of either $+1$ or $-1$. Let us denote a microstate of the system as a set of $N$ spins indexed by $i$, written

$$\nu = \{\sigma_i\}_{i=1}^N. \tag{14.17}$$

The energy function for this model system is

$$E(\{\sigma_i\}) = -h \sum_{i=1}^N \sigma_i - J \sum_{j \in \mathcal{N}(i)} \sigma_i \sigma_j \tag{14.18}$$

here $h$ is the external field and $J > 0$ is the magnitude of the attractive spin-spin coupling. The notation $\mathcal{N}(i)$ means the set of nearest neighbors on the lattice for spin $i$.

If we want to compute the Helmholtz free energy for this system, we can first attempt to compute the canonical partition function $Z$ and use the fact that

$$Z(\beta) = e^{-\beta A}. \tag{14.19}$$

Writing out the full expression for the partition function, we have

$$\begin{aligned} Z(\beta) &= \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} \cdots \sum_{\sigma_N=\pm 1} e^{-\beta E\left(\{\sigma_i\}_{i=1}^N\right)}, \\ &= \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} \cdots \sum_{\sigma_N=\pm 1} \exp\left(\beta h \sum_{i=1}^N \sigma_i + \beta J \sum_{j \in \mathcal{N}(i)} \sigma_i \sigma_j\right). \end{aligned} \tag{14.20}$$

In the past in similar situations, we have computed that partition function by factorizing the energy function into contributions from each degree of freedom. Here, we cannot do that because of the coupling between nearest neighbors. Next time we will develop techniques for approximating $Z$ for the Ising model.

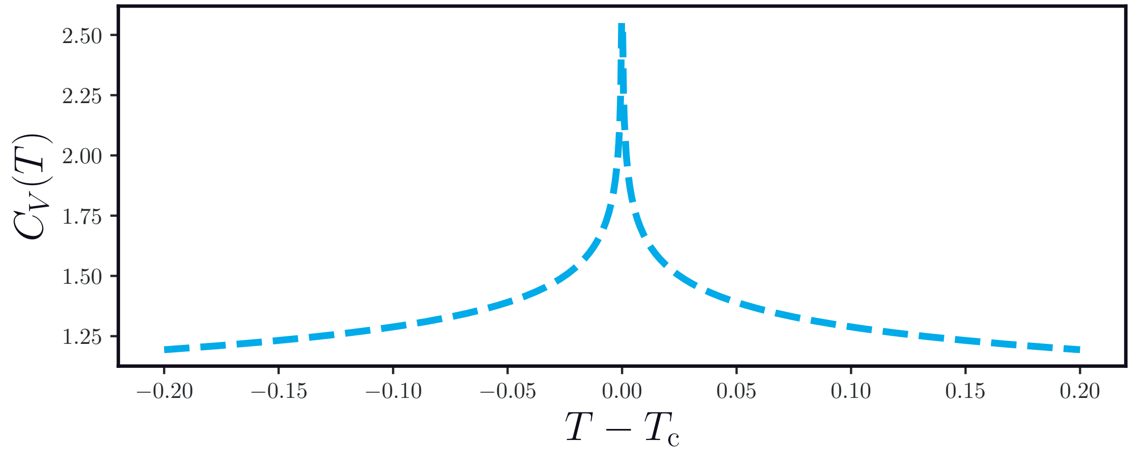


Figure 14.6: The characteristic scaling of the heat capacity near the critical temperature.

# Lecture 15 Ising model and mean-field theory

## Recap

1. We discussed the phenomenology of phase transitions, highlighting how, as one cools a magnet, there is *spontaneous symmetry breaking*. This, in turn, leads to large fluctuations near the critical point.

2. We introduced the Ising model, a lattice model that captured coupling to an external field $h$ and interactions between nearest neighbors of magnitude $J$. This model has the *non-factorizable* energy function
$$E_{\text{Ising}}(\sigma_1, \dots, \sigma_N) = -h \sum_{i=1}^{N} \sigma_i - J \sum_{i,j \in \mathcal{N}(i)} \sigma_i \sigma_j \tag{15.1}$$

## Goals for today

1. Establish a connection between the Ising magnet and the "lattice gas" model to highlight how simple physical considerations lead to universal critical behavior. Liquid-vapor transitions are governed by the same physics as the Ising magnet.

2. Compute the partition function for the 1d Ising model analytically.

3. Introduce "mean-field theory" to approximately compute the average magnetization in $d \geq 2$.

## 15.1 Universality motivates simple models

In the last class, we looked at the scaling of magnetization with respect to temperature near the critical temperature. We noted that
$$|\langle M \rangle| \propto |T - T_{\text{c}}^{\text{mag}}|^{b_{\text{mag}}}, \tag{15.2}$$
which is shown in Fig. 15.1, and
$$|\rho - \rho_{\text{c}}| \propto |T - T_{\text{c}}^{\text{liq}}|^{b_{\text{liq}}}. \tag{15.3}$$
both followed the same power law experimentally,
$$b_{\text{liq}} = b_{\text{mag}} \equiv \text{"}\beta\text{"}. \tag{15.4}$$
I put the name of this exponent in quotes to emphasize that this "$\beta$" is not the inverse temperature, but rather the name of a *critical exponent*. For a large variety of microscopically distinct materials,
$$\beta = 0.32. \tag{15.5}$$

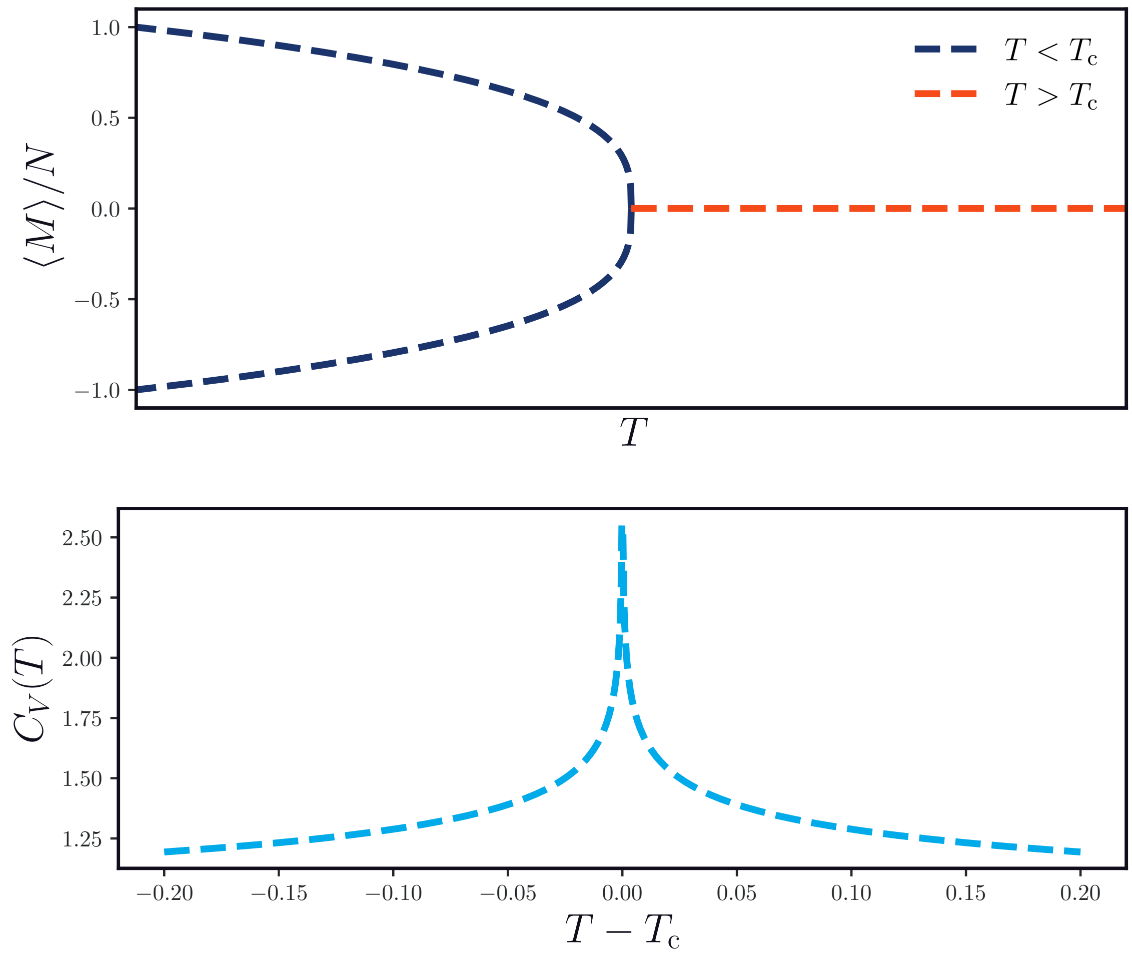


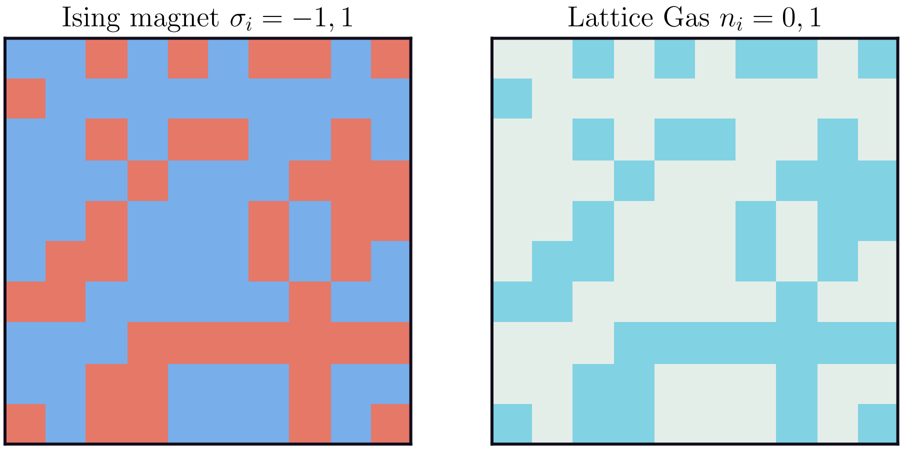
Figure 15.1: Scaling of the average magnetization near the critical temperature and the characteristic scaling of the heat capacity near the critical temperature.

The heat capacity diverges with a power law that is similarly universal,

$$C_\mathrm{V} \propto |T - T_\mathrm{c}|^{-\alpha}, \quad \alpha = 0.11. \tag{15.6}$$

The experimental reality of universality indicates that microscopic details are not particularly important for quantifying the behavior of systems near phase transitions. With this in mind, we introduced a very minimal model of a magnet, the Ising model. As shown in Fig. 15.2, it is a matter of perspective whether this model represents a magnet with spins $\sigma_i = \pm 1$ or a lattice gas where each site is an occupancy variable $n_i = \{0, 1\}$. Within the lattice gas model, attractive interactions can be represented by nearest neighbor couplings

$$E_\mathrm{LG}(n_1, \ldots, n_N) = -\epsilon \sum_{i=1}^{N} n_i n_j, \tag{15.7}$$


Ising magnet $\sigma_i = -1, 1$

Lattice Gas $n_i = 0, 1$

Figure 15.2: Isomorphism between the Ising magnet and the lattice gas model.

where $\epsilon$ is the magnitude of the interaction. We can then write the grand canonical partition function

$$\Xi_{\text{LG}} = \sum_{n_1=0,1} \cdots \sum_{n_N=0,1} \exp\left( \beta\epsilon \sum_{i,j\in N(j)} n_i n_j + \beta\mu \sum_{i=1}^{N} n_i \right). \tag{15.8}$$

This partition function should look very familiar because it is quite similar to the expression we wrote last time for the canonical partition function of the Ising magnet,

$$Z_{\text{m}} = \sum_{n_1=-1,1} \cdots \sum_{n_N=-1,1} \exp\left( \beta J \sum_{i,j\in N(j)} \sigma_i \sigma_j + \beta h \sum_{i=1}^{N} \sigma_i \right). \tag{15.9}$$

With a clever substitution, you can convert between $\sigma_i$ and $n_i$ to prove that these two are actually the same model (you will do this on the homework).

## 15.2 Solving the 1d Ising model

Last time we said that the Ising model is difficult to solve because the energy function involves spins that are coupled to each other. As a result, we cannot factorize the statistical weights that appear in the canonical partition function. However, in one-dimension ($d = 1$), the partition function is simple enough that we can still compute it. The dimensionality of a lattice model on a square lattice *only* affects the number of nearest neighbors.[*]

Let's set $h = 0$ and write the energy function for the Ising model,

$$E_{\text{Ising}}(\sigma_1, \ldots, \sigma_N) = -J(\sigma_1\sigma_2 + \cdots + \sigma_{N-1}\sigma_N). \tag{15.10}$$

---

[*] It should be easy to convince yourself that the number of neighbors $z = 2d$.

Because each $\sigma_i$ can only take on the values of $\pm 1$, we can just introduce a new variable

$$b_i := \sigma_i \sigma_{i+1}. \tag{15.11}$$

If we know $\{\sigma_1, b_1, \dots, b_{N-1}\}$ then we can compute $\{\sigma_1, \dots, \sigma_N\}$. The canonical partition function is thus

$$\begin{aligned} Z(\beta, h=0) &= \sum_{\sigma_1} \sum_{b_1} \cdots \sum_{b_{N-1}} e^{\beta J \sum_{i=1}^{N} b_i}, \\ &= 2 \sum_{b_1} \cdots \sum_{b_{N-1}} e^{\beta J \sum_{i=1}^{N} b_i}, \\ &= 2 \left( e^{\beta J} + e^{-\beta J} \right)^{N-1}, \\ &= 2^N (\cosh \beta J)^{N-1}. \end{aligned} \tag{15.12}$$

When $N$ is very large, we can approximate this as

$$Z(\beta, h=0) \underset{N \to \infty}{\approx} 2^N (\cosh \beta J)^N. \tag{15.13}$$

If this model has a phase transition as $N \to \infty$, we should expect diverging correlations. That is, we should compute

$$\left( \frac{\partial^2 \log Z}{\partial \beta^2} \right)_{N,V} = k_{\mathrm{B}} T^2 C_v(T). \tag{15.14}$$

Using the fact that the derivative of log cosh is tanh, we obtain

$$\begin{aligned} \left( \frac{\partial^2 \log Z}{\partial \beta^2} \right)_{N,V} &= NJ \frac{\partial}{\partial \beta} \tanh \beta J \\ &= NJ^2 (1 - (\tanh \beta J)^2) \end{aligned} \tag{15.15}$$

This function has no divergence as shown in Fig. 15.3, which means there is no phase transition at finite temperature for the 1d Ising model. Why not?

The reason is closely related to the dimensionality. When $d = 1$ the cost of creating an interface between domains of up spins and down spins is always $2J$. That is,

$$\Delta E_{\text{interface},1d} = 2J. \tag{15.16}$$

However, each domain increases the entropy; there are $N$ choices for where to put an interface so

$$T \Delta S_{\text{interface},1d} = k_{\mathrm{B}} T \log N. \tag{15.17}$$

In the limit $N \to \infty$, the entropic term will dominate and the probability of having an interface (breaking the long-range order) will go to 1. On the other hand, in $d = 2$ the cost of forming an interface scales like $\sqrt{N}$.

$$\Delta E_{\text{interface},2d} = 2J\sqrt{N}. \tag{15.18}$$

There are only $2\sqrt{N}$ choices for where to put an interface so

$$T \Delta S_{\text{interface},2d} = k_{\mathrm{B}} T \log 2\sqrt{N}. \tag{15.19}$$

Now, in the limit $\Delta A = \Delta E - T \Delta S \to \infty$, meaning an interface will never form.

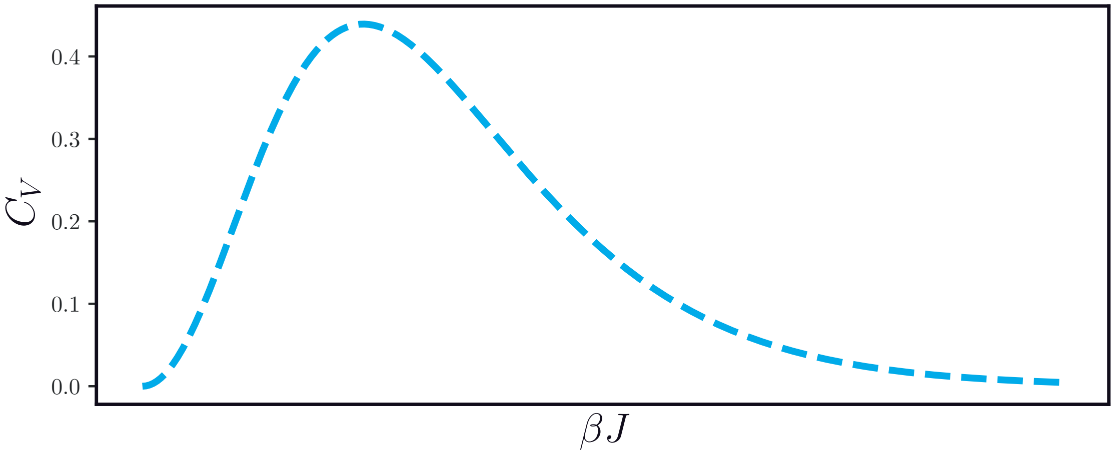


Figure 15.3: The heat capacity for the 1d Ising model.

## 15.3 Mean-field approximation

To decouple the spins in $d \geq 2$, we replace each neighbor with its average magnetization. As such, we can write

$$E(\sigma_1, \ldots, \sigma_N) = -\sum_{i=1}^{N} h_{\text{eff}}(i)\sigma_i \tag{15.20}$$

where

$$h_{\text{eff}}(i) = h + J \sum_{j \in \mathcal{N}(i)} \sigma_j . \tag{15.21}$$

Because $h_{\text{eff}}$ is not statistically independent of $\sigma_i$, if we want to factorize, we can replace each of the neighbors by its average $m = \langle \sigma_j \rangle$ . That is, we replace the interaction term with a *mean-field*

$$h_{\text{mf}} = h + Jzm \tag{15.22}$$

where $z$ again denotes the number of nearest neighbors (and hence is $2d$ on a square lattice). Within this approximation, we can write,

$$\begin{aligned} Z_{\text{mf}} &= \left( \sum_{\sigma=\pm 1} e^{\beta h_{\text{mf}} \sigma} \right)^N \\ &= \left( 2\cosh(\beta h + \beta Jzm) \right)^N . \end{aligned} \tag{15.23}$$

This partition function depends on the average magnetization, a quantity that we may not know *a priori*. However, we can determine the average magnetization by taking a derivative of $\log Z$; in other words

$$m = \frac{1}{N} \left( \frac{\partial \log Z_{\text{mf}}}{\partial \beta J z} \right)_{N,V} = \tanh(\beta h + \beta Jzm). \tag{15.24}$$

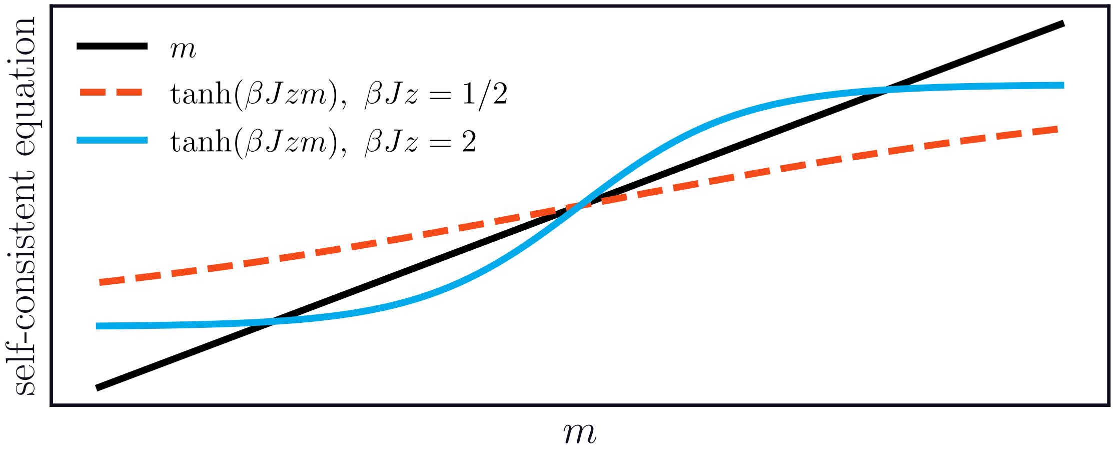


Figure 15.4: Self-consistent equations from mean-field theory show spontaneous symmetry breaking in $d = 1$ (A wrong prediction! There's no $d = 1$ phase transition...) and $d = 2$, where there is in fact a phase transition.

The average magnetization appears on both the right and left-hand sides of this equation. Such a relation is called a *self-consistent mean-field equation*. A solution for $m$ is the point where the line $y = m$ intersects the curve $y = \tanh(\beta h + \beta Jzm)$. These curves are plotted in Fig. 16.1 for $h = 0$. When $\beta Jz \leq 1$, that is, the temperature is high, the curves intersect only at $m = 0$. When $\beta Jz > 1$, there is *spontaneous symmetry breaking*, and non-zero average magnetization can occur. In other words, mean-field theory predicts a critical point at $k_B T_c = Jz$.

# Lecture 16 Markov chain Monte Carlo

## Recap

1. Discussed the "lattice gas" model and its connection with the Ising model.
2. Computed the partition function for the 1d Ising model analytically and obtained
$$Z_{\text{Ising},1d}(\beta) = 2^N \cosh^N(\beta J). \tag{16.1}$$
We saw that there was no phase transition at finite $\beta$, which we deduced by showing that the heat capacity had no divergence.
3. We introduced the idea of "mean-field theory", which gave us an approximate expression for the partition function. Solving a *self-consistent* equation for the average magnetization led us to deduce the existence of a phase transition.

## Goals for today

1. Assess the accuracy of mean-field theory.
2. Discuss the role of computer simulations in *sampling* statistical ensembles.
3. Introduce an ubiquitous simulation technique, Markov Chain Monte Carlo.

## 16.1 Mean-field approximation

To decouple the spins in $d \geq 2$, we replace each neighbor with its average magnetization. As such, we can write
$$E(\sigma_1, \dots, \sigma_N) = -\sum_{i=1}^{N} h_{\text{eff}}(i)\sigma_i \tag{16.2}$$
where
$$h_{\text{eff}}(i) = h + J \sum_{j\in\mathcal{N}(i)} \sigma_j. \tag{16.3}$$

Because $h_{\text{eff}}$ is not statistically independent of $\sigma_i$, if we want to factorize, we can replace each of the neighbors by its average $m = \langle\sigma_j\rangle$ . That is, we replace the interaction term with a *mean-field*
$$h_{\text{mf}} = h + Jzm \tag{16.4}$$

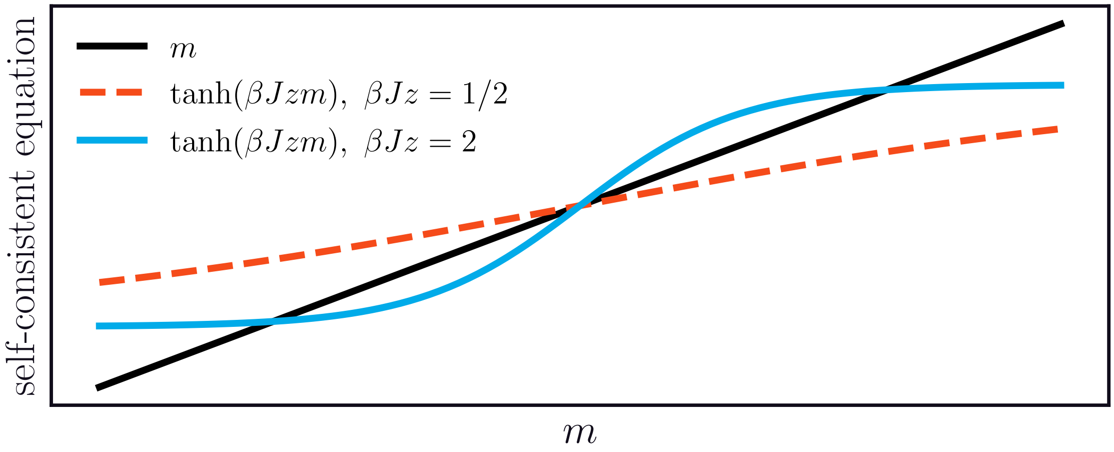


Figure 16.1: Self-consistent equations from mean-field theory show spontaneous symmetry breaking in $d = 1$ (A wrong prediction! There's no $d = 1$ phase transition...) and $d = 2$, where there is in fact a phase transition.

where $z$ again denotes the number of nearest neighbors (and hence is $2d$ on a square lattice). Within this approximation, we can write,

$$\begin{aligned} Z_{\text{mf}} &= \left( \sum_{\sigma=\pm1} e^{\beta h_{\text{mf}} \sigma} \right)^N \\ &= \left(2\cosh(\beta h + \beta Jzm)\right)^N . \end{aligned} \tag{16.5}$$

The prefactor accounts for double counting in the partition function (though we typically neglect it because it is an additive constant in the free energy and has no impact on the average magnetization). This partition function depends on the average magnetization, a quantity that we may not know *a priori*. However, we can determine the average magnetization by taking a derivative of $\log Z$; in other words

$$m = \frac{1}{N} \left( \frac{\partial \log Z_{\text{mf}}}{\partial \beta h_{\text{mf}}} \right)_{N,V} = \tanh(\beta h + \beta Jzm). \tag{16.6}$$

The average magnetization appears on both the right and left-hand sides of this equation. Such a relation is called a *self-consistent mean-field equation*. A solution for $m$ is the point where the line $y = m$ intersects the curve $y = \tanh(\beta h + \beta Jzm)$. These curves are plotted in Fig. 16.1 for $h = 0$. When $\beta Jz \leq 1$, that is, the temperature is high, the curves intersect only at $m = 0$. When $\beta Jz > 1$, there is *spontaneous symmetry breaking*, and non-zero average magnetization can occur. In other words, mean-field theory predicts a critical point at $k_B T_c = Jz$.

How accurate is mean-field theory? A remarkable observation here is that mean-field theory is exact for $d \geq 4$. We say that $d = 4$ is the upper critical dimension of the Ising model.

| $d$ | $T_c$ (exact) | $k_\mathrm{B}T_c$ (MFT) |
|---|---|---|
| $d=1$ | $0$ | $2J$ |
| $d=2$ | $2.269J$ | $4J$ |
| $d=3$ | $4.5J$ | $6J$ |
| $d\geq 4$ | $2dJ$ | $2dJ$ |

Table 16.1: Summary of the accuracy of mean-field theory as a function of the dimension.

## 16.2 Computer simulations

The mean-field approximation is a rather brutal one (and there are more intricate techniques that give more accurate answers, like the renormalization group). However, the best option in many cases is to avoid approximation all together by relying on computer simulations. Computation is an *essential* part of modern statistical mechanics; for complex, interacting systems it provides us with the tools to accurately estimate properties of many distinct systems.

What precisely should we compute? Thus far, when we have encountered a new system, we have attempted to compute the partition function. Let us consider the following code snippet.

```
"""
function to compute the partition function.
input: n, side length of Ising model (n*n total spins)
"""
def compute_Z_ising2d(n):
  Z = 0
  for i in range(2**(n*n)):
    lattice = get_conf(i)
    Z += energy(lattice)
  return Z
```

The function looks plausible, but it will not work. Even for a small model $10\times 10$ spins, the loop will require $2^{100}\approx 10^{30}$, an impossibly large number of iterations.[*]

A better idea is to compute averages of observables. Let us consider an observable $f(\nu)$, a function that maps a configuration of the system to a number or vector with relevant information about that configuration. For example, one observable for the Ising model could be the average magnetization,

$$f(\sigma_1,\ldots,\sigma_N)=\frac{1}{N}\sum_{i=1}^{N}\sigma_i,\quad \nu=\{\sigma_i\}_{i=1}^{N}. \tag{16.7}$$

If we want to compute an average value for this quantity, we need to estimate

$$\begin{aligned}\langle f\rangle &= \int_\nu f(\nu)p(\nu)d\nu,\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(\nu_i),\quad \nu_i\sim p.\end{aligned} \tag{16.8}$$

---

[*] A simple way of making this estimate: $2^{10}=1024\approx 10^3$.

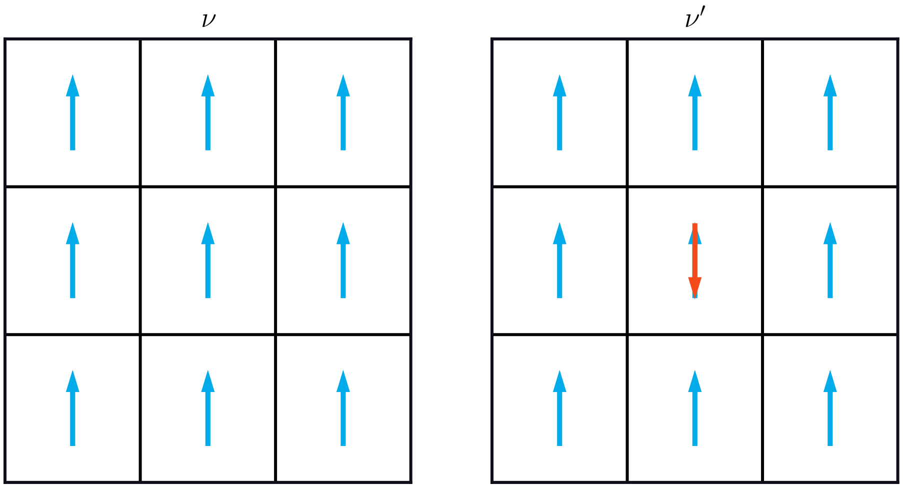


Figure 16.2: A single spin flip proposal move from an MCMC algorithm.

where $p$ is the probability of a microstate, which could be given by the canonical, microcanonical, grand canonical, or isothermal-isobaric ensemble. We are back to the original problem, however, because

$$p_{\text{canonical}}(\nu) = e^{-\beta E(\nu)}/Z(\beta) \tag{16.9}$$

and we *do not* know $Z$.

### 16.2.1 Markov Chain Monte Carlo

We need a procedure for collecting configurations in proportion to their statistical mechanical probabilities, a procedure we call *sampling*. Markov chain Monte Carlo (MCMC) algorithms are an essential tool for this task. Monte Carlo methods are algorithms that exploit random number generators to generate randomly distributed states. As shown in Fig. 16.2, the proposed configuration randomly selects a spin and flips it. However, we need a mechanism to decide if the newly generated configuration is statistically likely. In practice, we do this via "rejection sampling".

At a high level, most MCMC algorithms implement the following type of procedure:

Markov Chain Monte Carlo

1. Start with initial state $\nu$
2. Generate random perturbed state $\nu + \Delta\nu \equiv \nu'$
3. Evaluate the relative probability of the new state $\nu'$ to the current state $\nu$.

$$\frac{p(\nu')}{p(\nu)} = \frac{e^{-\beta E(\nu')}/Z}{e^{-\beta E(\nu)}/Z} = e^{-\beta(E(\nu')-E(\nu))}. \tag{16.10}$$

4. Accept or reject the proposed configuration $\nu'$ with probability

$$p_{\text{acc}}(\nu \to \nu') = \min[1, e^{-\beta\Delta E(\nu,\nu')}] \tag{16.11}$$

Note that this procedure means if we generate a configuration with lower energy than the current configuration, we *always* accept it. This procedure generates a sequence (or a chain) of configurations $\nu_1, \ldots, \nu_K$ that we can use to estimate average values of observables. Importantly, if we propose $\nu'$ from $\nu_i$ and reject it, then we *must* set $\nu_{i+1} = \nu_i$. This is how, mathematically, the statistical likelihood of $\nu_i$ is emphasized.

The property that we need to ensure that we are sampling the correct target distribution is called detailed balance. Mathematically, it says that

$$p(\nu)p(\nu \to \nu') = p(\nu')p(\nu' \to \nu) \tag{16.12}$$

where the transition probability consists of both the probability of generating the new configuration and the probability of accepting, denoted $p_{\text{gen}}$ and $p_{\text{acc}}$, respectively. That is,

$$p(\nu \to \nu') = p_{\text{gen}}(\nu \to \nu')p_{\text{acc}}(\nu \to \nu'). \tag{16.13}$$

Why do we need this property at all? Global detailed balance ensures that the Markov Chain is reversible, which will mean that there are no net flows of probability in the stationary distribution. This ensures that the equilibrium distribution is sampled.

Hence, what we need to ensure for procedure outlined above is that with our acceptance criterion, we satisfy detailed balance. We must check

$$\begin{aligned}\frac{p(\nu)}{p(\nu')} = e^{-\beta\Delta E} &= \frac{p_{\text{gen}}(\nu' \to \nu)p_{\text{acc}}(\nu' \to \nu)}{p_{\text{gen}}(\nu \to \nu')p_{\text{acc}}(\nu \to \nu')}, \\ &= \frac{\min[1, e^{-\beta\Delta E}]}{\min[1, e^{+\beta\Delta E}]}.\end{aligned} \tag{16.14}$$

Reasoning through the final expression, if $\Delta E$ is positive, we get the desired result because the denominator is 1. If $\Delta E$ is negative, then in the denominator we get $1/e^{-\beta\Delta E}$, which again, produces the desired result! Therefore, the "Metropolis" acceptance criterion

$$p_{\text{acc}}(\nu \to \nu') = \min[1, e^{-\beta\Delta E}] \tag{16.15}$$

satisfies detailed balance.

One last matter that we need to specify regarding computer simulations. We need to accept or reject the proposed configurations with a specified probability, but how, in practice, do we actually sample an event with that given probability? There is a simple trick that we employ: because pseudo-random number generators can be used to produce random numbers in the interval $[0,1]$, the full range of possible values for a probability. Because these numbers are uniformly distributed, the probability that a randomly generated number lies in the interval $[0,p]$ is simply $p$. We can use this fact to easily implement the Metropolis acceptance criterion.

Metropolis acceptance criterion

```
p_acc = np.min(1, np.exp(-beta*dE))
random_event = np.random.rand()
if random_event < p_acc:
  return accept
```

## Lecture 17 Models of the liquid state

## Recap

1. Introduced the Markov Chain Monte Carlo (MCMC) algorithm, a procedure that generates a sequence or "chain" of states
$$\nu_0, \nu_1, \dots, \nu_K \tag{17.1}$$
such that for any reasonable observable $f$
$$\langle f \rangle = \sum_{\nu} f(\nu)p(\nu) = \lim_{K\to\infty} \frac{1}{K}\sum_{i=1}^{K} f(\nu_i). \tag{17.2}$$
2. Defined the Metropolis (Rosenbluth[*])-Hastings acceptance criterion,
$$p_{\text{acc}}(\nu \to \nu') = \min[1, e^{-\beta \Delta E(\nu,\nu')}]. \tag{17.3}$$
3. Discussed the principle of detailed balance,
$$p(\nu)p(\nu \to \nu') = p(\nu')p(\nu' \to \nu), \tag{17.4}$$
where
$$p(\nu \to \nu') = p_{\text{gen}}(\nu \to \nu')p_{\text{acc}}(\nu \to \nu'). \tag{17.5}$$

## Goals for today

1. Generalize MCMC to "off-lattice" systems with continuous state spaces.
2. Introduce the "hard disk" potential energy and discuss MCMC in this model.
3. Discuss the partition function for classical liquids.

---

[*] Arianna Rosenbluth passed away in 2021 at the age of 93. Her career and contributions were nicely summarized in this obituary in the New York Times. Someone should talk to the headline writer, though—chemical physics became "data science".

## 17.1 General purpose Monte Carlo algorithm

Markov Chain Monte Carlo Last time we introduced a Monte Carlo algorithm in terms of the Ising model. We considered proposal moves where the new state was generated using a spin flip. However, the algorithm we introduced only requires that we can compute energy differences after generating a new state.

1. Start with initial state $\nu$
2. Generate random perturbed state $\nu + \Delta\nu \equiv \nu'$
3. Evaluate the relative probability of the new state $\nu'$ to the current state $\nu$.

$$\frac{p(\nu')}{p(\nu)} = \frac{e^{-\beta E(\nu')}/Z}{e^{-\beta E(\nu)}/Z} = e^{-\beta(E(\nu')-E(\nu))}. \tag{17.6}$$

4. Accept or reject the proposed configuration $\nu'$ with probability

$$p_{\text{acc}}(\nu \to \nu') = \min[1, e^{-\beta\Delta E(\nu,\nu')}] \tag{17.7}$$

To reiterate again[*], the two properties we need are *ergodicity*, meaning that we visit every state, and *detailed balance* which ensures we sample the states in relative proportion to the target distribution.

We often use MCMC methods to study interacting particle systems because, like the Ising model, the partition functions for these systems do not factorize. Among the simplest interactions is volume exclusion, which can be modeled with a hard repulsive wall, as shown in Fig. 17.1. We write the potential energy for the system in terms of a pairwise interaction potential, which specifies the energy of the interaction between particles $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. This pairwise interaction is almost always isotropic, meaning it only depends the distance between the particles, which we denote

$$r_{ij} = \|\boldsymbol{x}_i - \boldsymbol{x}_j\|. \tag{17.8}$$

For hard disks, the pairwise interaction is

$$u(r) = \begin{cases} 0 \text{ if } r > r_{\text{hs}}, \\ \infty \text{ otherwise.} \end{cases} \tag{17.9}$$

The total energy of the system is then

$$U(\boldsymbol{x}^N) = \sum_{i<j} u(r_{ij}) \tag{17.10}$$

where $\boldsymbol{x}^N = \{\boldsymbol{x}_1, \dots \boldsymbol{x}_N\}$. Summing over $i < j$ ensures that we count each pair of particles exactly once. This system makes for a simple MCMC model, because we don't really need to compute an energy difference. If we generate a configuration in which two particles overlap, we simply reject it.

---

[*] intentional redundancy

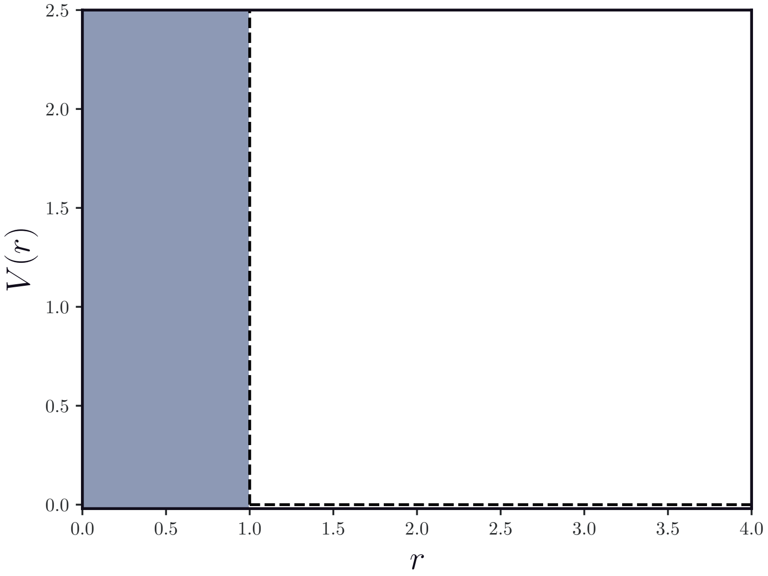


Figure 17.1: A potential for hard disks.

## 17.2 Radial distribution functions

Hard disks are only subtly different from an ideal gas. The only correlations that we obtain arise from the volume occupied by the particles. How does the equation of state for hard disks deviate from the ideal gas law? For an ideal gas, the density is spatially uniform because

$$\rho = \frac{N_\mathrm{A}}{V} = \frac{p}{RT}. \tag{17.11}$$

To quantify the deviation from this uniform density is useful to introduce a radial distribution function, which quantifies the density around a tagged particle. First, we need to count the number of particles in a window around the tagged particle. We do this computing the number of particles in an infinitesimally thin ring or "annulus" around the tagged particle and then normalizing by the volume of the annulus. That is, we define the radial distribution function as

$$g(r) = \frac{n(r)}{4\pi\rho r^2 dr} \tag{17.12}$$

where $\rho$ is the bulk density. The "$g$ of $r$", as we often call it, is dimensionless and asymptotically approaches 1. Because the $g(r)$ tells us the probability of finding two particles separated by a distance $r$, we can approximate

$$g(r) \approx e^{-\beta u(r)}, \tag{17.13}$$

which is the Boltzmann weight for a configuration with two particles separated by $r$.

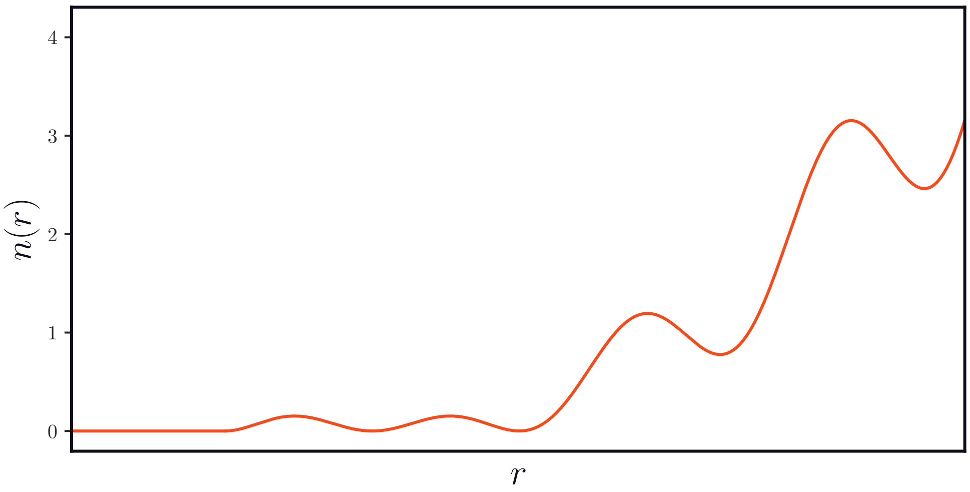


Figure 17.2: Number of particles in annulus with inner radius $r$ and outer radius $r + dr$.

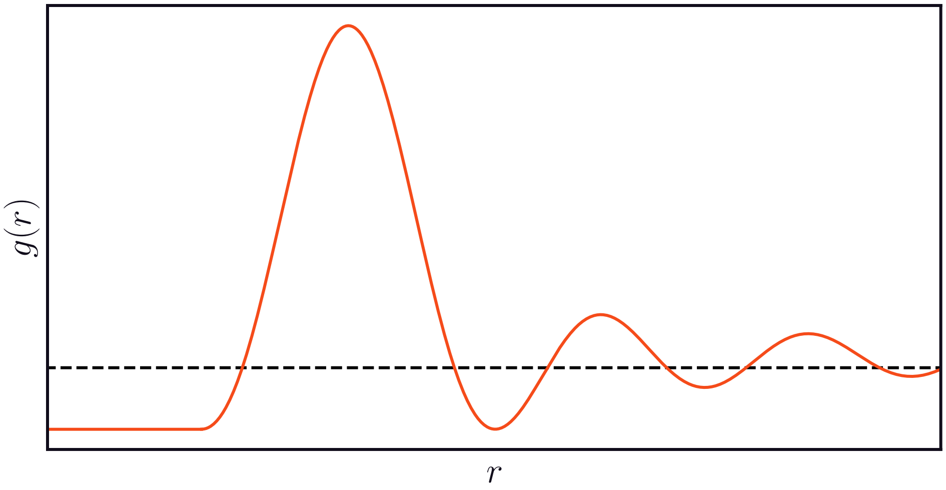


Figure 17.3: The radial distribution function for the particle density in annulus with inner radius $r$ and outer radius $r + dr$. The black dashed line is the bulk density.

## 17.3 Partition function for classical liquids, virial expansion

A classical microstate is fully specified by $\boldsymbol{x}^N, \boldsymbol{p}^N$, the positions and momenta. Unlike in quantum mechanics, we can know both simultaneously. The generic Hamiltonian for an interacting particle system is

$$\mathcal{H}(\boldsymbol{x}^N, \boldsymbol{p}^N) = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m} + U(\boldsymbol{x}^N). \tag{17.14}$$

One nice property of the kinetic energy part of the Hamiltonian is that it always factorizes*. Integrating out the quadratic contribution, we obtain the following expression for the canonical partition function

$$Z(\beta) = \frac{1}{N!\lambda_{\mathrm{T}}^{3N}} \int_V e^{-\beta U(\boldsymbol{x}^N)} d\boldsymbol{x}^N. \tag{17.15}$$

Using this expression, we can compute the pressure to examine how the equation of state compares to the ideal gas equation of state. For a non-ideal system,

$$\rho \neq \frac{p}{RT} \tag{17.16}$$

but can we correct the equation of state? Using the fact that

$$A = -\beta^{-1} \log Z(\beta) \tag{17.17}$$

and the thermodynamic derivative

$$\left(\frac{\partial A}{\partial V}\right)_{N,T} = -p \tag{17.18}$$

we can look at how interactions change the pressure.

We have a somewhat tricky expression in (18.6) to deal with, however. Let us start by writing the exponential as a product over all pairs of particles in the system,

$$e^{-\beta U(\boldsymbol{x}^N)} = \prod_{i<j} e^{\beta u(r_{ij})}. \tag{17.19}$$

We want to obtain an expression that is going to tell us about deviations from ideality, so we rewrite this as

$$e^{-\beta U(\boldsymbol{x}^N)} = \prod_{i<j} 1 + f_{ij} \text{ with } f_{ij} \equiv e^{-\beta u(r_{ij})} - 1. \tag{17.20}$$

Expanding this product

$$e^{-\beta U(\boldsymbol{x}^N)} = 1 + \sum_{i<j} f_{ij} + \sum_{i<j} \sum_{k<l} f_{ij} f_{kl} + \dots \tag{17.21}$$

---

*see Quiz #1

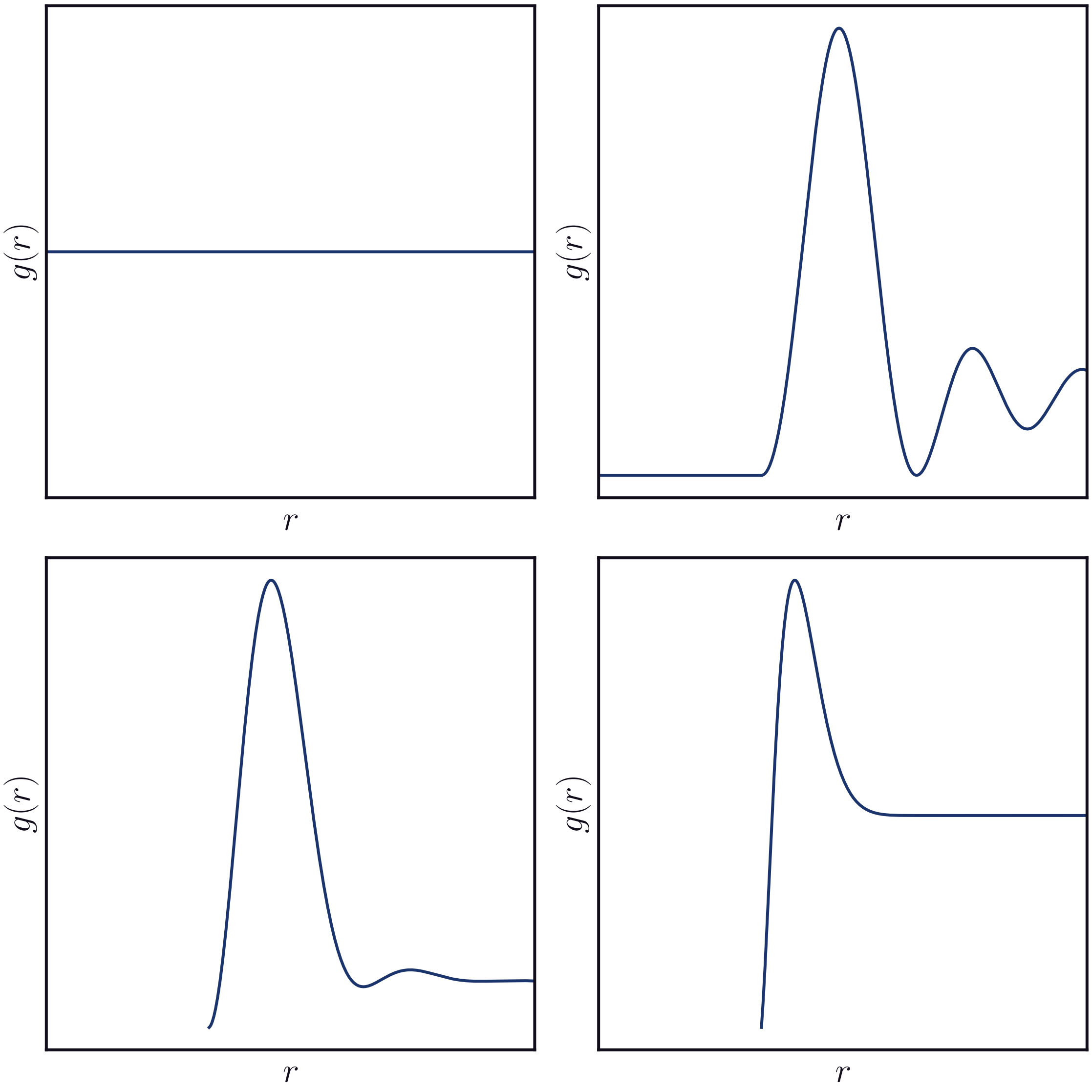


Figure 17.4: Comparison of schematic representations (i.e., not real data) of the $g(r)$ for (clockwise from top left) an ideal gas, a solid, a liquid, and a real gas.

Putting this expression into the definition of $Z$, we see that

$$\begin{aligned} Z(\beta) &= \frac{1}{N!\lambda_{\mathrm{T}}^{3N}} \int_V (1 + \sum_{i<j} f_{ij} + \dots) d\boldsymbol{x}^N \\ &= \frac{1}{N!\lambda_{\mathrm{T}}^{3N}} \left( V + \int_V (\sum_{i<j} f_{ij} + \dots) d\boldsymbol{x}^N \right) \\ &= \frac{1}{N!\lambda_{\mathrm{T}}^{3N}} \left( V + \frac{N(N-1)}{2} \int_V f d\boldsymbol{x} \right) \\ &= \frac{V}{N!\lambda_{\mathrm{T}}^{3N}} \left( 1 + \frac{N(N-1)}{2V} \int_V f d\boldsymbol{x} \right). \end{aligned} \tag{17.22}$$

The first term on the RHS is the ideal gas partition function, $Z_{\text{ideal}}$.

Now, taking the derivative of $A = -\beta^{-1} \log Z$ with respect to the volume, $V$, we get (minus) the pressure,

$$\begin{aligned} \beta p &= \rho - \rho^2 \int_V f(r) d\boldsymbol{r} + \mathcal{O}(\rho^3), \\ &= \rho + \rho^2 \int_V 1 - e^{-\beta u(r)} d\boldsymbol{r} + \mathcal{O}(\rho^3) \\ &= \rho + B(T)\rho^2 + \mathcal{O}(\rho^3) \end{aligned} \tag{17.23}$$

The quantity $B(T)$ is known as the first virial coefficient. Next time we will compute it exactly for hard disks.

## Lecture 18 Classical molecular dynamics

## Recap

1. Introduced the hard disk model, where the pairwise interaction potential was given by

$$u(r) = \begin{cases} 0 \text{ if } r < r_{\text{cut}} \\ \infty \text{ otherwise} . \end{cases} \tag{18.1}$$

2. Discussed the radial distribution function, a central object in liquid state theory, defined as

$$g(r) = \frac{n(r)}{4\pi\rho r^2 dr}, \tag{18.2}$$

which at low density, we can approximate in terms of the Boltzmann distribution

$$g(r) \approx e^{-\beta u(r)}. \tag{18.3}$$

3. Wrote an expression for the pressure of a non-ideal system,

$$\beta p = \rho + B(T)\rho^2 + \mathcal{O}(\rho^3), \tag{18.4}$$

which we saw to be

$$B(T) = 2\pi \int_V (1 - g(r))r^2 dr, \tag{18.5}$$

where the factor of $2\pi$ comes from converting to radial coordinates.

## Goals for today

1. Introduce classical molecular simulation.
2. Introduce the "Lennard-Jones" model of van der Waals interactions.
3. Write the classical energy function in a many-body expansion.

## 18.1 Partition function for classical systems, virial expansion

A classical microstate is fully specified by $\boldsymbol{x}^N, \boldsymbol{p}^N$, the positions and momenta. Integrating out the quadratic contribution, we obtain the following expression for the canonical partition function

$$Z(\beta) = \frac{1}{N!\Lambda_{\mathrm{T}}^{3N}} \int_V e^{-\beta U(\boldsymbol{x}^N)} d\boldsymbol{x}^N. \tag{18.6}$$

Once we have access to the partition function, we can easily compute thermodynamic properties. This is due to the fact that the partition function is related to a thermodynamic potential,

$$A = -\beta^{-1} \log Z \tag{18.7}$$

so the pressure is

$$\beta p = -\beta \left(\frac{\partial A}{\partial V}\right)_{N,T} = \left(\frac{\partial \log Z}{\partial V}\right)_{N,T}. \tag{18.8}$$

The ideal gas equation of state tell us that $p/k_{\mathrm{B}}T = N/V$, but adding interactions will alter this expression for the pressure. In the last lecture, we expanded the partition function (18.6) to obtain

$$\begin{aligned} \beta p &= \rho - \frac{1}{2}\rho^2 \int_V f(\boldsymbol{r}) d\boldsymbol{r} + \mathcal{O}(\rho^3), \\ &= \rho + \frac{1}{2}\rho^2 \int_V 1 - e^{-\beta u(\boldsymbol{r})} d\boldsymbol{r} + \mathcal{O}(\rho^3) \\ &= \rho + B(T)\rho^2 + \mathcal{O}(\rho^3) \end{aligned} \tag{18.9}$$

The quantity $B(T)$ is known as the first virial coefficient. Using the approximation

$$g(r) \approx e^{-\beta u(r)} \tag{18.10}$$

and writing this expression in terms of the one dimensional radial distance $r$,

$$B(T) = 2\pi \int_0^\infty (1 - g(r)) r^2 dr \tag{18.11}$$

**Exercise!**

For the hard disk model, you can easily evaluate $B(T)$ using the approximation $g(r) \approx e^{-\beta u(r)}$. Do this calculation and derive an expression for $p/k_{\mathrm{B}}T$ with corrections up to order $\rho^2$.

## 18.2 Molecular dynamics

As we have seen, it is not typically possible to compute the partition function

$$Z(\beta) = \frac{1}{N!\Lambda_T^{3N}} \int_V e^{-\beta U(\boldsymbol{x}^n)} d\boldsymbol{x}^N \tag{18.12}$$

when the interaction potential $U$ does not factorize into single particle contributions. Markov chain Monte Carlo simulations give statistical insight into the system by sampling

$$p(\boldsymbol{x}^N) \propto e^{-\beta U(\boldsymbol{x}^N)} \tag{18.13}$$

from which we can compute averages (or higher cumulants) of any observable property

$$\begin{aligned} \langle f \rangle &= Z^{-1}(\beta) \int_V f(\boldsymbol{x}) e^{-\beta U(\boldsymbol{x})} d\boldsymbol{x} \\ &\approx \frac{1}{K} \sum_i^K f(\boldsymbol{x}_i) \quad \text{with } \boldsymbol{x}_i \sim e^{-\beta U}/Z. \end{aligned} \tag{18.14}$$

Due to ergodicity, we know that ensemble averages like the one written above can be replaced with dynamical averages through correlated trajectories. That is,

$$\begin{aligned} \langle f \rangle &= Z^{-1}(\beta) \int_V f(\boldsymbol{x}) e^{-\beta U(\boldsymbol{x})} d\boldsymbol{x} \\ &= \lim_{t \to \infty} \frac{1}{t} \int_0^t f(\boldsymbol{x}_t) dt. \end{aligned} \tag{18.15}$$

General overview of Molecular Dynamics

1. Generate initial structure (x-ray crystallography, alphafold).
2. Evaluate the energy / forces of the system.
3. Propagate the system.
4. Compute observables.

Let us break down each of these, step by step.

**Initialization.** Choosing the initial configuration and velocities can be either a complicated or a simple task. For proteins with no experimentally known structure, one must rely on homology models and structure prediction tools. For a homogeneous liquid at a given target density, typically we simply put the particles on a lattice. Velocities are drawn from a Maxwell-Boltzmann distribution (a Gaussian) at the target temperature,

$$p(\boldsymbol{v}_i) \propto e^{-\beta m \boldsymbol{v}^2/2} \tag{18.16}$$

**Energy of a classical system.** There is one highly accurate (though still approximate) way to compute the total energy. Within the Born-Oppenheimer approximation, we solve

$$\mathcal{H}_e \Psi_{e,i} = E_{e,i} \Psi_{e,i} \tag{18.17}$$

which is the electron Schrödinger equation on nuclear coordinate $i$. Once $\Psi_{e,i}$ is determined the forces on the nuclei can be computed via the Hellman-Feynman theorem. Denoting the nuclear coordinates $\boldsymbol{x}^N$, recall that the force is just

$$f(\boldsymbol{x}^N) = -\nabla_{\boldsymbol{x}} E(\boldsymbol{x}^n). \tag{18.18}$$

Obviously, solving (18.17) at each configuration visited during a dynamical simulation is extremely expensive. Algorithms based on this procedure are known as *ab initio* molecular dynamics.

Classical molecular dynamics attempts to avoid this costly evaluation of the energy by developing an approximate energy function (often called a "classical forcefield") to approximate the interactions without using electronic structure methods. While this approach does not allow us to describe important quantum mechanical effects (electronic excitations, bond-breaking), for many systems these effects do not play a role in determining material properties.

**Propagating the system.** We have already seen how to sample new configurations using MCMC. Molecular dynamics samples the Boltzmann distribution using a fundamentally different approach that relies on a physical model of how the system propagates. We first compute a vector of forces

$$\vec{f}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = -\left(\nabla_{\boldsymbol{x}} E(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)\right)_i \tag{18.19}$$

and subsequently solve the equation of motion for Newtonian dynamics

$$\ddot{\boldsymbol{x}}_i = m_i^{-1} \vec{f}_i. \tag{18.20}$$

This gives us a deterministic equation of motion!

**Computing observables.** Just as in MCMC, we can estimate ensemble averages using the principle of ergodicity. For a given observable $A(\boldsymbol{x})$,

$$\langle A \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t A(\boldsymbol{x}_t) dt. \tag{18.21}$$

Both in MCMC and MD, the configurations that are collected will be statistically correlated. In practice, this means that one generally subsamples a trajectory, spacing configurations by some duration $\tau$, and computes

$$\langle A \rangle \approx \frac{1}{K} \sum_{i=0}^{K-1} A(\boldsymbol{x}_{i\tau}) \tag{18.22}$$

## 18.3 Building a classical forcefield

Our approximate energy function must capture the essential physical contributions to the interactions between atoms. If the length scales we are interested in are sufficiently long and temperatures are sufficiently high, then electronic quantum effects will be minimal. As a result, we can approximate an explicit representation of the electronic degrees of freedom by a classical charge distribution. We can use a *many-body expansion* to represent the most general function that describes an arbitrary interaction among a set of atoms

$$U(\boldsymbol{x}^N) = \sum_i v_i(\boldsymbol{x}_i) + \sum_{i<j} v_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j) + \sum_{i<j<k} v_{ijk}(\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k) + \ldots. \tag{18.23}$$

In practice, it is rare to use three-body interactions because they often make only a minor difference to the physics of the model and they scale poorly with the system size. The pairwise interaction requires computing $N^2$ pairwise interactions, three-body interactions require $N^3$ terms, which quickly becomes prohibitively large. One exceptional case where three-body terms are quite important is polarization phenomena: in this case, if atom $i$ polarizes atom $j$, then the strength of the interaction between $j$ and other nearby atoms $k$ depends on the position of $i$.

The general strategy of building a molecular forcefield is capture all the essential physical contributions. Let's start with an important one—van der Waals or London dispersion forces. This force leads to a weak attraction over short distances and is due to dipole-dipole coupling. The distance dependence of the strength of this coupling can be derived by a second order perturbation theory,

$$V_{\mathrm{vdW}} \propto -r^{-6}. \tag{18.24}$$

At very short ranges, there is also a repulsion (this is due to Pauli exclusion). Experimentally, the distance dependence of that repulsion is approximately

$$V_{\mathrm{excl}} \propto r^{-12}. \tag{18.25}$$

This is also computationally convenient because $(r^6)^2 = r^{12}$. Lennard-Jones[*] put these two ingredients together and introduced the widely-used Lennard-Jones potential. This pair potential has a simple functional form,

$$U_{LJ}(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma^{6}}{r}\right)\right], \tag{18.26}$$

where $\epsilon$ sets the depth of the minimum and $\sigma$, where the potential intersects zero, is often interpreted as the radius of a particle.

This potential can be parameterized to give experimentally accurate descriptions of low temperature noble gases, where van der Waals forces predominate. However, to propagate the system, we need to solve

$$\ddot{\boldsymbol{x}}^N = \vec{f}/\vec{m} \tag{18.27}$$

which is an ordinary differential equation. Next time, we will describe how to do this numerically.

---

[*] Though his name makes many people think otherwise, John Jones was a single person. In quite a progressive gesture at the time, he married Kathleen Lennard in 1925 and became John Lennard-Jones.

## Lecture 19 Hamiltonian dynamics

## Recap

1. Discussed *molecular dynamics* simulation. The key idea was to develop a classical energy function $U(\boldsymbol{x}^N)$ that would provide approximate interatomic forces for a system of interacting particles.
2. Described the notion of an "observable" $A$ and discussed how, if the dynamics is ergodic, we can estimate equilibrium ensemble averages using time trajectories. That is,

$$\begin{aligned} \langle A \rangle &= \int_V A(\boldsymbol{x}) e^{-\beta U(\boldsymbol{x})} d\boldsymbol{x} \\ &= \lim_{t \to \infty} \frac{1}{t} \int_0^t A(\boldsymbol{x}_t) dt. \end{aligned} \tag{19.1}$$

3. Wrote down Newton's second law,

$$\ddot{\boldsymbol{x}}_i = f_i(\boldsymbol{x}^N)/m_i = -\nabla_i U(\boldsymbol{x}^N), \tag{19.2}$$

   where the force $f_i$ is just the $i$th component of the gradient of the potential energy.

4. Introduced the Lennard-Jones potential, an empirical description of van der Waals interactions,

$$U_{LJ}(r) = 4\epsilon \left[ \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^{6} \right]. \tag{19.3}$$

## Goals for today

1. Discuss a generic MD simulation workflow.
2. Write down Hamilton's equations and describe how to integrate them numerically.
3. Introduce the Velocity-Verlet algorithm which is used to propagate the configuration by solving Hamilton's equations numerically.

## 19.1 Molecular dynamics as a flowchart

In the previous lecture, we introduced a basic workflow that we used to set up and carry out a molecular dynamics simulation. The main ingredients were 1) an initial configuration or structure of the molecule or system of interest 2) a model for the energy function ("forcefield") for the system

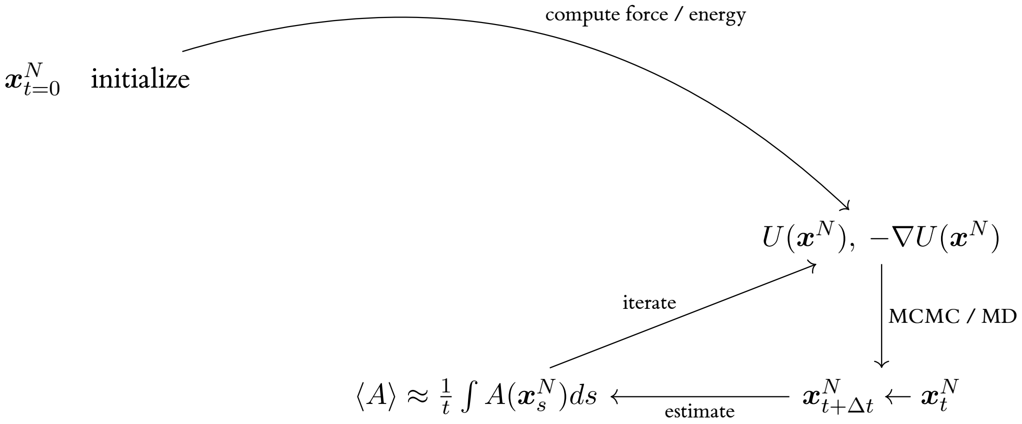


Figure 19.1: Generic MD simulation procedure.

3) a set of dynamical equations of motion for the atoms of the system and 3) a quantity of interest to measure. This procedure is summarized in the diagram below in Fig. 19.1.

A somewhat more procedural view is presented in the code block below. You will implement the force calculation to run a Lennard-Jones simulation on the homework.

```
"""
simple molecular dynamics implementation
"""
xs, vs = initialize() # some function to initialize

def lj_force(xs, epsilon=1.0, sigma=1.0):
  force = np.zeros([n, n, 2])
  for i in range(n):
      for j in range(i+1, n):
          r = compute_distance(xs[i], xs[j])
          force[i,j] = # TODO (homework!)
          force[j,i] = -force[i,j]
  return force

def run_traj(xs, vs, n_steps, dt=1e-4):
    traj = []
    # compute the initial forces
    f, e = lj_force_energy(xs)
    for step in range(n_steps):
        xs, vs, f, e = integrate_step(xs, vs, force, dt=dt)
        traj.append(xs.copy())
    return traj
```

Last time we discussed doing a many-body expansion; in this procedure we wrote an expression for the energy in terms of interactions with the other particles in the system. In classical systems, we can often neglect many-body interactions and approximate the energy of the system in terms of a pairwise interation,

$$U(\boldsymbol{x}^N) \approx \sum_{i<j} u(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|). \tag{19.4}$$

We make this approximation not only because the three-body terms are not physically important in most cases, but also because the computational cost scales poorly with system size if we do not truncate the many-body expansion. At each timestep, we must compute the force on each particle $i$, which we do by computing the gradient

$$f_i(\boldsymbol{x}^N) = -\nabla_i \left( \sum_{j\neq i}^{N} u(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|) \right). \tag{19.5}$$

For each particle, this requires an operation that scales like $N$ total distance computations. Because there are $N$ particles in the system, the force calculation has an overall scaling of

$$\text{cost to compute } f(\boldsymbol{x}^N) \sim \mathcal{O}(N^2) \text{ operations.} \tag{19.6}$$

This is the most expensive part of a classical molecular dynamics simulation.

## 19.2 Propagating the configuration

Once we have determined the forces, we still need to solve the ordinary differential equation

$$\ddot{\boldsymbol{x}}_i = f_i(\boldsymbol{x}^N)/m_i, \tag{19.7}$$

where the notation $\ddot{a}$ means the second time derivative $\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}$. Let us proceed by first noting that the velocity, the first time of the position, can be related to the momentum vector via the ordinary differential equation

$$\dot{\boldsymbol{x}}_i = \boldsymbol{p}_i/m_i. \tag{19.8}$$

Of course, we also know that

$$\dot{\boldsymbol{p}}_i = m_i \ddot{\boldsymbol{x}}_i = f_i(\boldsymbol{x}^N), \tag{19.9}$$

by Newton's second law (19.7). Writing the force on particle $i$ in terms of a derivative of the potential, we have

$$\dot{\boldsymbol{p}}_i = f_i(\boldsymbol{x}^N) = -\frac{\partial U}{\partial \boldsymbol{x}_i} \tag{19.10}$$

Let us keep this relation front of mind. The classical Hamiltonian is given by the sum of the kinetic energy and the potential energy

$$\mathcal{H}(\boldsymbol{x}^N, \boldsymbol{p}^N) = \mathcal{K}(\boldsymbol{p}^N) + U(\boldsymbol{x}^N) \tag{19.11}$$

where

$$\mathcal{K}(\boldsymbol{p}^N) = \frac{1}{2} \sum_{i=1}^{N} \boldsymbol{p}_i^2/m_i \tag{19.12}$$

and

$$U(\boldsymbol{x}^N) = \sum_{i<j} u(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|). \tag{19.13}$$

Notice that equation for the momentum (19.10) is determined entirely by the potential energy function $U$. That is, we could just as well have written the differential equation in terms of the Hamiltonian because the derivative does not interact with the kinetic energy contribution,

$$\dot{\boldsymbol{p}}_i = f_i(\boldsymbol{x}^N) = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}_i}. \tag{19.14}$$

Conveniently, we can also write the evolution for the particle position as a differential equation involving the Hamiltonian. From the definition of the velocity (19.8), we see that

$$\dot{\boldsymbol{x}}_i = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i} = \boldsymbol{p}_i/m_i \tag{19.15}$$

because the only term that contributes to the derivative is the term that is quadratic in $\boldsymbol{p}_i$ in $\mathcal{K}(\boldsymbol{p}^N)$.

Together, these coupled equations of motion are known as Hamilton's equations. We write the equation of motion in terms of these first order differential equations not only because the formulation is elegant[*] but also because it makes some physical properties of this dynamics more transparent.

$$\dot{\boldsymbol{x}}_i = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i} \qquad \dot{\boldsymbol{p}}_i = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}_i}. \tag{19.16}$$

To demonstrate this, let us ask the question: how does the total energy change as a function of when the system evolves according to (20.1)? We must compute

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}(\boldsymbol{x}^n, \boldsymbol{p}^N)}{\mathrm{d}t} &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i}\dot{\boldsymbol{p}}_i + \frac{\partial \mathcal{H}}{\partial \boldsymbol{x}_i}\dot{\boldsymbol{x}}_i \\ &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i}\left(-\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}_i}\right) + \frac{\partial \mathcal{H}}{\partial \boldsymbol{x}_i}\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i}\right) \qquad = 0. \end{aligned} \tag{19.17}$$

Hamiltonian dynamics conserves energy. That means if we apply this dynamics, we will sample a distribution with fixed $N, V, E$. In other words, Hamiltonian dynamics is the dynamics of the microcanonical ensemble.

## 19.3 Numerical integration

To solve Hamilton's equations on a computer, we first discretize the problem. That is, we write the time derivatives as finite differences computed over a small time step $\Delta t$. We can think of this discretization like doing a Taylor expansion,

$$\boldsymbol{x}^N(t+\Delta t) = \boldsymbol{x}^N(t) + \dot{\boldsymbol{x}}^N(t)\Delta t + \frac{1}{2}\ddot{\boldsymbol{x}}^N(t)\Delta t^2 + \mathcal{O}(\Delta t^3). \tag{19.18}$$

---

[*] and it is!

At this point, though, we have other expressions for the time derivatives of the position and we could have equivalently written

$$\boldsymbol{x}^N(t+\Delta t) = \boldsymbol{x}^N(t) + \frac{\boldsymbol{p}^N(t)}{\boldsymbol{m}}\Delta t + \frac{f(\boldsymbol{x}^N(t))}{2\boldsymbol{m}}\Delta t^2 + \mathcal{O}(\Delta t^3). \tag{19.19}$$

Because we also have an equation of motion for $\boldsymbol{p}$ we can use that directly to update the momenta,

$$\boldsymbol{p}^N(t+\frac{\Delta t}{2}) := \boldsymbol{p}^N(t) + f(\boldsymbol{x}^N(t))\frac{\Delta t}{2}, \tag{19.20}$$

which we can then use to update the positions (using a midpoint evaluation of $\boldsymbol{p}^N$),

$$\boldsymbol{x}^N(t+\Delta t) := \boldsymbol{x}^N(t) + \boldsymbol{m}^{-1}\boldsymbol{p}^N(t+\Delta t/2)\Delta t, \tag{19.21}$$

and, after recomputing the force, we advance the momenta again by a half-step

$$\boldsymbol{p}^N(t+\Delta t) := \boldsymbol{p}^N(t+\Delta t/2) + f(\boldsymbol{x}^N(t+\Delta t))\frac{\Delta t}{2}. \tag{19.22}$$

Why split things up like this? The main reason is numerical accuracy. Notice that simply plugging in our expression for the momenta at the intermediate time point, we get

$$\boldsymbol{x}^N(t+\Delta t) := \boldsymbol{x}^N(t) + \left[\boldsymbol{m}^{-1}\boldsymbol{p}^N(t) + \boldsymbol{m}^{-1}f(\boldsymbol{x}^N(t))\frac{\Delta t}{2}\right]\Delta t, \tag{19.23}$$

which is exactly the Taylor expansion at order $\Delta t^2$. This implies a high order of local accuracy in the numerical scheme.

# Lecture 20 Molecular dynamics at constant temperature

## Recap

1. We formulated Hamiltonian dynamics

$$\dot{\boldsymbol{x}}_i = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i} \qquad \dot{\boldsymbol{p}}_i = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}_i}. \tag{20.1}$$

   where

$$H(\boldsymbol{x}^N, \boldsymbol{p}^N) = \mathcal{K}(\boldsymbol{p}^N) + U(\boldsymbol{x}^N) = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m_i} + \sum_{i<j} u(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|). \tag{20.2}$$

   We also proved (via a simple calculation) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\boldsymbol{x}^N, \boldsymbol{p}^N) = 0 \tag{20.3}$$

   when $\boldsymbol{x}^N$ and $\boldsymbol{p}^N$ evolve according to (20.1). That is, Hamiltonian dynamics conserves energy.

2. We introduced the velocity Verlet algorithm to solve the coupled differential equations (20.1). This algorithm numerically integrates the ODEs using a "symplectic" integrator. At each step, we advance the position and momenta as

$$\begin{aligned} \boldsymbol{p}^N(t + \frac{\Delta t}{2}) &:= \boldsymbol{p}^N(t) + f(\boldsymbol{x}^N(t))\frac{\Delta t}{2}, \\ \boldsymbol{x}^N(t + \Delta t) &:= \boldsymbol{x}^N(t) + \boldsymbol{m}^{-1}\boldsymbol{p}^N(t + \Delta t/2)\Delta t, \\ \boldsymbol{p}^N(t + \Delta t) &:= \boldsymbol{p}^N(t + \Delta t/2) + f(\boldsymbol{x}^N(t + \Delta t))\frac{\Delta t}{2}. \end{aligned} \tag{20.4}$$

   Though it splits the momenta update into two steps, approach achieves a high level of local accuracy compared to a non-symplectic integrator, and it is worth the computational effort because there is still only one force calculation per position update.

## Goals for today

1. Discuss molecular dynamics in the canonical ensemble and introduce the idea of the Andersen thermostat.
2. Discuss using molecular dynamics to obtain *dynamical information*.
3. Discuss the impact of dynamical fluctuations on reaction kinetics and provide a short introduction to transition state theory.

## 20.1 Sampling canonical configurations with molecular dynamics

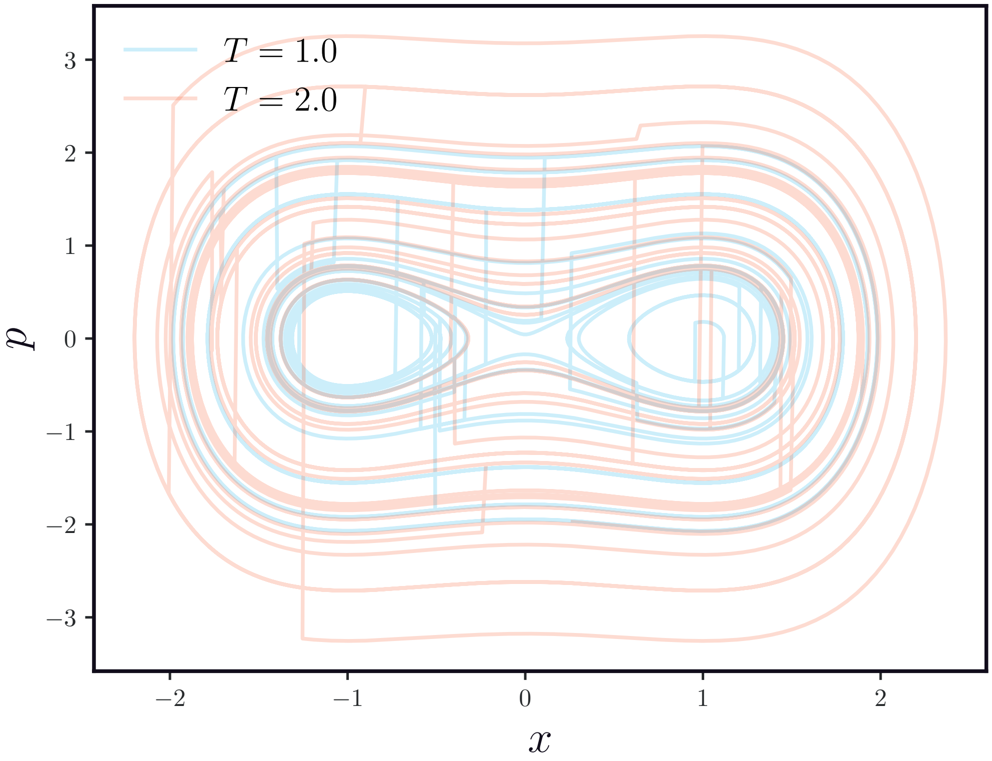


Figure 20.1: Solution to the ODEs (20.1), i.e., the *phase space trajectories* of $x_t, p_t$ for two different initial conditions where $U(x) = (1 - x^2)^2$ is a bistable potential energy function.

For many chemical systems, we would prefer to sample at constant temperature as opposed to constant energy. To carry this out, we need to conserve a quantity related to the temperature because the infinite thermal reservoir that we conceptualized in thermodynamics does exist in our simulation. The natural way to define the temperature in a molecular dynamics simulation is to relate it to the average kinetic energy. The Maxwell-Boltzmann distribution tells us that the probability of a momentum vector $\boldsymbol{p}_i$ is

$$\text{Prob.}(\boldsymbol{p}_{i,x}) = Z_{\text{kin}}(\beta)^{-1} e^{-\beta \boldsymbol{p}_{i,x}^2/2m_i}. \tag{20.5}$$

This probability distribution is one that we know well, a Gaussian:

$$\text{Prob.}(\boldsymbol{p}_i) = \mathcal{N}(\mu, \sigma), \quad \mu = 0 \quad \sigma = \sqrt{m_i k_{\text{B}} T}. \tag{20.6}$$

If we compute the average kinetic energy per degree of freedom,

$$\langle \boldsymbol{p}_i^2 / 2m_{i,x} \rangle = \frac{1}{Z_{\text{kin}}} \int \boldsymbol{p}_{i,x}^2 e^{-\beta \boldsymbol{p}_{i,x}^2/2m_i} d\boldsymbol{p}_{i,x}, \tag{20.7}$$

we have the nice result that this is just the variance $\sigma^2$ because the mean is zero. Thus,

$$\frac{1}{2m_i}\left\langle \boldsymbol{p}_{i,x}^2 \right\rangle = \frac{1}{2m_i}\sigma^2 = \frac{1}{2}k_{\mathrm{B}}T. \tag{20.8}$$

That is, for each classical degree of freedom, the momentum contributes $1/2k_{\mathrm{B}}T$ to the overall energy. This result is typically called the *classical equipartition of energy*.

The average kinetic energy is, of course, easy to measure in a classical molecular dynamics simulation because we know all the at each time step momenta, so we can average

$$\langle \mathcal{K} \rangle = \frac{1}{N}\sum_{i=1}^{N}\left\langle \boldsymbol{p}_i^2 \right\rangle /2m_i \equiv \frac{3}{2}Nk_{\mathrm{B}}T. \tag{20.9}$$

In other words, the effective instantaneous temperature of the system is

$$T_{\text{eff}}(\boldsymbol{p}^N) = \frac{2}{3k_{\mathrm{B}}N^2}\sum_{i=1}^{N}\boldsymbol{p}_i^2/2m_i. \tag{20.10}$$

To preserve this property on average, we introduce the “Andersen thermostat.” This method was developed by Stanford chemistry emeritus professor Hans C. Andersen [*] The central idea of the Andersen thermostat is to re-sample the velocities so that the average kinetic energy is consistent with a Maxwell-Boltzmann distribution. The frequency with which this resampling is performed is determined by a stochastic model—a particle has a change in momentum if it collides with a “bath particle”. We do not explicitly represent the bath degrees of freedom, but we assume there is an independent probability of such a collision for each particle in the system with rate $\gamma$. The probability distribution that describes these collisions is a Poisson distribution, which has probability density function

$$p(t;\gamma) = \gamma e^{-\gamma t}. \tag{20.11}$$

In other words, the probability of a collision in the interval $[t, t+\Delta t]$ is $p(t;\gamma)\Delta t$, which for sufficiently small $\Delta t$ is

$$\text{Prob.(collision)} \approx \gamma \Delta t, \tag{20.12}$$

after a Taylor expansion. This formulation leads to a simple, elegant algorithm for constant temperature simulation.

---

[*] No, not that Hans C. Andersen... here’s the original paper.

Andersen Thermostat

```
"""
pseudocode implementation of the Andersen thermostat
t_max: total simulation time
gamma: friction coefficient
"""
while t < t_max:
  # run a step of dynamics
  xs_t, vs_t = velocity_verlet_step(xs_t, vs_t, dt)

  # compute the probability of a collision
  # between a particle and the bath
  p_collide =  gamma * dt

  if np.random.rand() < p_collide:
    vs_t = maxwell_boltzmann_vs(beta, ms) # resample the velocities
```

At this point, the dynamics is no longer deterministic; the Andersen thermostat makes the simulation a Markov process that samples the canonical distribution. That is, we again have a distribution of states that rigorously samples the Boltzmann distribution in the long time limit.

## 20.2 Dynamical properties

The reintroduction of stochastic dynamics raises the question: are the deterministic Hamiltonian (NVE) dynamics useful for anything? One thing that determinism allows is that we can obtain *dynamical* information directly from simulations. For example, suppose we observed the mean-squared displacement of a particle as a function of time

$$\mathrm{MSD}(t) = \left\langle |x(t) - x(0)|^2 \right\rangle \tag{20.13}$$

Let us imagine the dynamics of a tagged particle in a liquid. The mean-squared displacement, at short times, is dominated by ballistic motion in which

$$\Delta x \propto v \Delta t \implies \mathrm{MSD} \propto t^2. \tag{20.14}$$

However, at longer time, there are collisions with other particles and the particle behaves more like a random walk. In homework #2, we derived the scaling,

$$\mathrm{MSD} \propto t. \tag{20.15}$$

Diffusive dynamics controls the rates of many chemical processes.

We have thought about reactions only in "mass-action" limit. We computed equilibrium constants in terms of the partition function,

$$K_{\text{eq}} = e^{-\frac{\Delta E_0}{RT}} \prod_j \left( \frac{(z_{j,0}^{\ominus})^{v_j}}{N_A} \right) \tag{20.16}$$

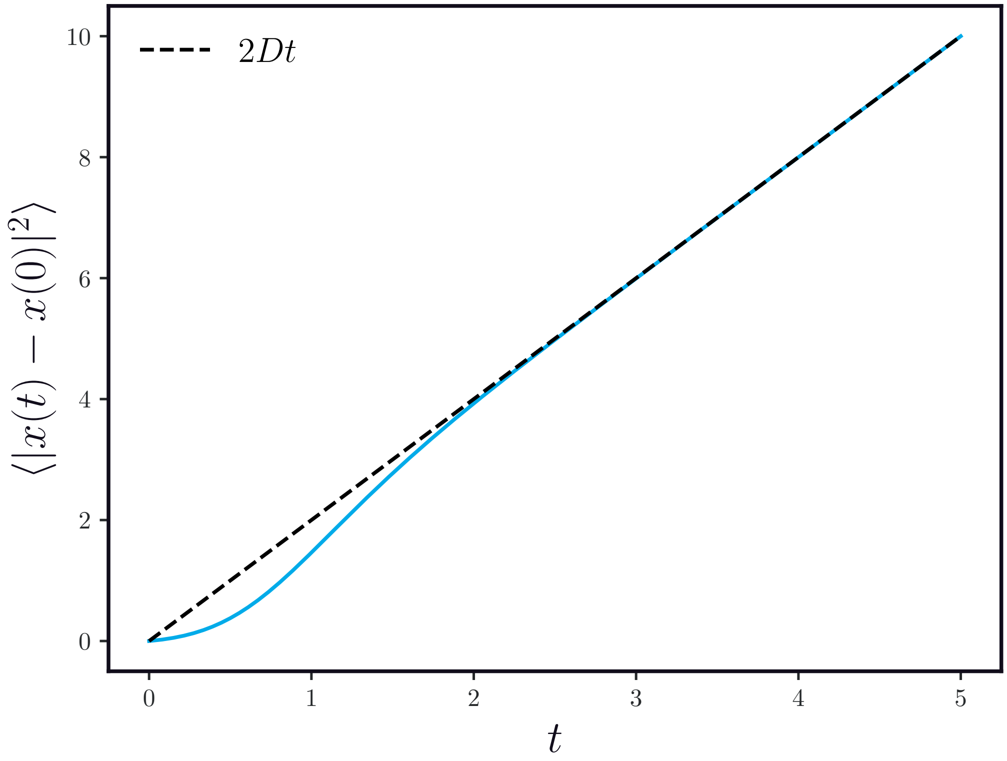


Figure 20.2: Schematic depiction of the mean-squared displacement as a function of time. At large times, the mean-squared displacement scales like $2Dt$ where $D$ is the diffusion coefficient.

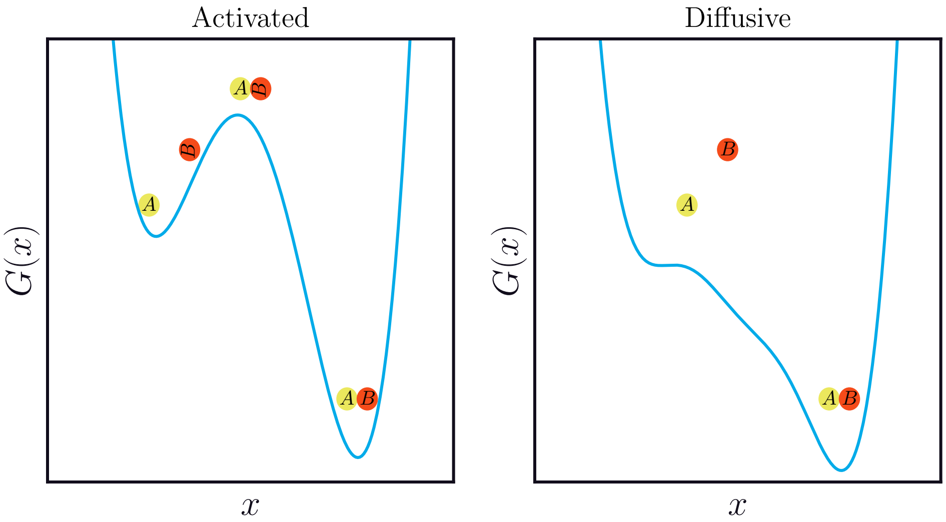


Figure 20.3: Two types of dimerizations. For an activated process, the diffusive step is not rate-limiting, rather some slow process that controls binding is the rate limiting step. For a diffusive process, the rate is entirely controlled by the probability that two particles interact.

but this does *not* reveal to us the pathway that connects the two. In fact, we can construct very simple examples where there is a profound difference in the rate of a process while keeping the equilibrium constant $K_{\text{eq}}$ fixed. We can think of a multistep process with a diffusion dominated rate $k_D$ and a separate rate for the activated process $k_a$. Over the next week, we will focus on obtaining statistical mechanical expressions for $k_D$ and $k_a$.

# Lecture 21 Collision theory

## Recap

1. We derived the *Classical Equipartition of Energy* which says that each degree of freedom contributes $1/2k_\mathrm{B}T$ to the overall kinetic energy. In other words, for a three-dimensional system consisting of $N$ classical particles,
$$\langle \mathcal{K} \rangle = \frac{3}{2} N k_\mathrm{B} T. \tag{21.1}$$
2. We introduced the Andersen thermostat, which maintained the average kinetic energy in a molecular dynamics simulation by stochastically rescaling the velocities to maintain $\langle K \rangle$. Each particle collides with a bath particle with probability
$$p_\text{collision}(\Delta t; \gamma) = \gamma e^{-\gamma \Delta t} \Delta t \approx \gamma \Delta t. \tag{21.2}$$
3. We started to discuss the impact of dynamics on the rate of a chemical process.

## Goals for today

1. This lecture will focus on diffusion limited processes, so we will try to compute the typical time before a collision between particles.
2. We will compute the mean free path length, the typical distance travelled without a collision.
3. We will then compute the rate constant $k_\mathrm{D}$.

## 21.1 Distribution of particle speeds

Last time, we considered two distinct types of processes, as depicted in Fig. 21.1. In the diffusion limited case, a robust prediction of the rate will require first that we known something about the typical speeds that particles are moving. We have, at this point, primarily written the Maxwell-Boltzmann distribution for the momenta. Of course, we can use this distribution to deduce the statistics of the speed $v$. First, let us note that

$$v = \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}_x^2 + \boldsymbol{v}_y^2 + \boldsymbol{v}_z^2}. \tag{21.3}$$

This equation emphasizes that are many velocity vectors $\boldsymbol{v}$ with the same speed $v$; they all correspond to the vectors of magnitude $v$ and hence they define a sphere of radius $v$. If we then compute

$$p(\boldsymbol{v}_x) = \frac{1}{\sqrt{2\pi k_\mathrm{B} T/m}} e^{-\beta m \boldsymbol{v}_x^2/2} \tag{21.4}$$

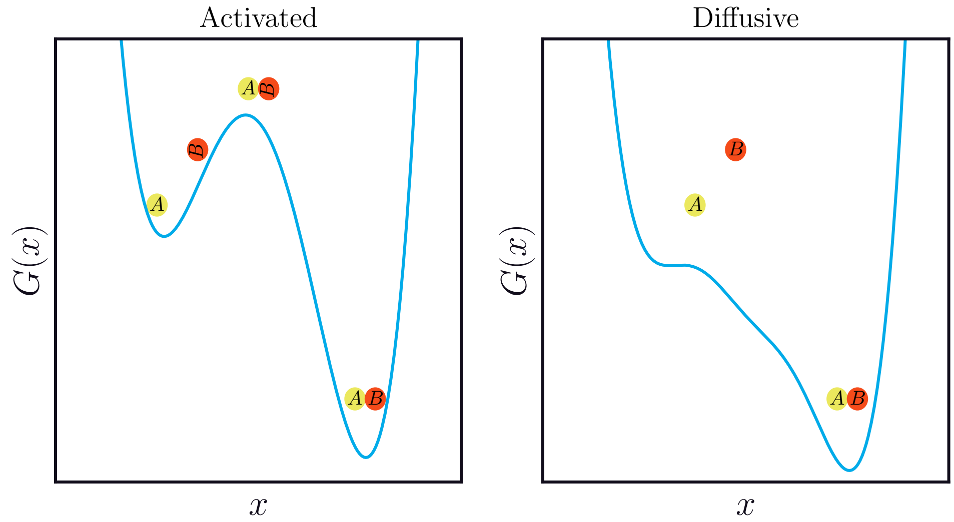


Figure 21.1: Two types of dimerizations. For an activated process, the diffusive step is not rate-limiting, rather some slow process that controls binding is the rate limiting step. For a diffusive process, the rate is entirely controlled by the probability that two particles interact.

which is a Gaussian distribution with mean zero and variance

$$\text{var}(\boldsymbol{v}_x) = k_\text{B}T/m. \tag{21.5}$$

This expression emphasizes that when the temperature is large, a particle of mass $m$ will typically have larger velocity fluctuations than the same particle at lower temperatures. What are the implications for the typical speeds? To compute the distribution of typical speeds, we compute

$$p(v) = p(\boldsymbol{v}) \times (\text{\# vectors with } \|\boldsymbol{v}\|^2 = v^2). \tag{21.6}$$

We know how to deal with both of the terms on the right hand side:

$$p(v) = \text{Maxwell-Boltzmann} \times \text{surface area of sphere } dv. \tag{21.7}$$

In three-dimensions, this is explicitly

$$p(v)dv = \left(\frac{m}{2\pi k_\text{B}T}\right)^{3/2} e^{-\beta m v^2/2} 4\pi v^2 dv. \tag{21.8}$$

The velocity and speed distributions are plotted in Fig. 21.2.

Once we have access to this distribution it is possible to compute the average speed of a particle.

$$\begin{aligned}\langle v\rangle &= \int_0^\infty v p(v) dv \\ &= \left(\frac{m}{2\pi k_\text{B}T}\right)^{3/2} \int_0^\infty 4\pi v^3 e^{-\beta m v^2/2} dv.\end{aligned} \tag{21.9}$$

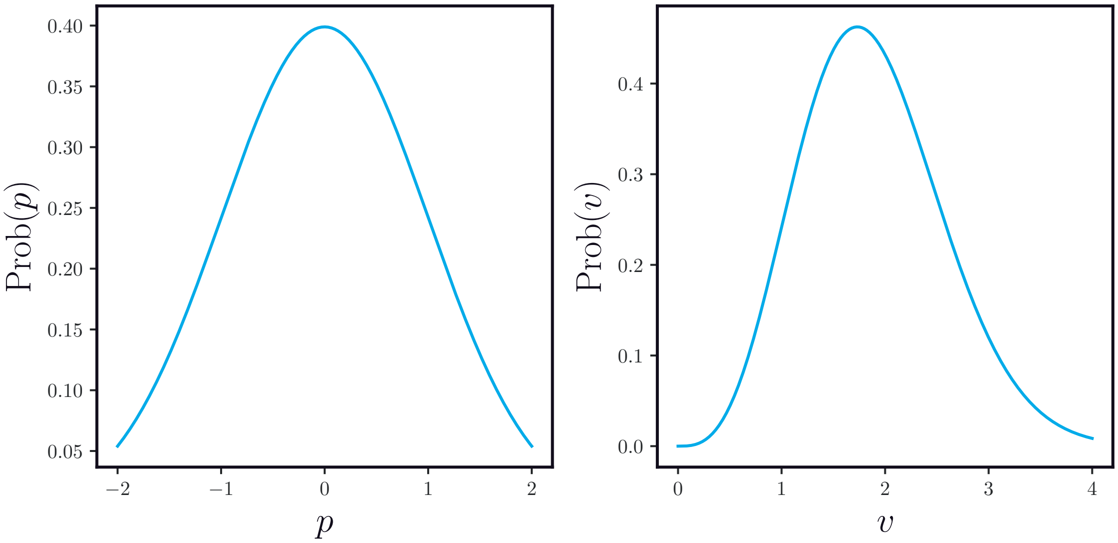


Figure 21.2: Velocity distribution and speed distribution

This is yet another Gaussian integral (which can be computed by differentiating inside the integral). Using the fact that

$$\int x^3 e^{-\alpha x^2} dx = \frac{1}{2\alpha^2}, \tag{21.10}$$

we see that

$$\langle v \rangle \equiv v_{\text{avg}} = \sqrt{\left(\frac{8k_{\text{B}}T}{\pi m}\right)}. \tag{21.11}$$

A nice example from Berg's book is provided by the enzyme lysozyme. Its mass is $2.3 \times 10^{-20}$ g and $k_{\text{B}}T = 4.1410^{-14}$ g · cm²/s² leads to an average speed of

$$\langle v \rangle_{\text{lysozyme}} = 21.41 \text{ m/s}. \tag{21.12}$$

For comparison, when Usain Bolt ran the world record in the 100m dash in 9.58s, he averaged a speed of $10.44$ m/s.

## 21.2 Collision probabilities

The speed distribution that we have just discussed neglects collisions. To simplify the discussion, assume that we have a system of homogeneous particles with radius $r$ in two-dimensions. A collision occurs whenever $d(\boldsymbol{x}_i, \boldsymbol{x}_j) < r$. To avoid a collision, must have a free area of diameter $d = 2r$ surrounding it that remains unoccupied by particles. That is, each particle needs

$$A_{\text{free}} = \pi d^2 \equiv \sigma. \tag{21.13}$$

In the dilute limit (when we can approximate the probability density of finding a particle a position $\boldsymbol{x}$ by the density $\rho(\boldsymbol{x})$), we have

$$p_{\text{collision}} = \sigma \rho d\boldsymbol{x}, \tag{21.14}$$

noting that the spatial homogeneity allows us to drop the $\boldsymbol{x}$-dependence of $\rho$.

We now analyze how the number of collisions in the system evolves as a function of the distance the particles have travelled. To keep track of the particles that are moving ballistically (i.e., without a collision), we introduce the quantity

$$n_{\text{free}}(\boldsymbol{x}) = \#\text{ of particles reaching } \boldsymbol{x} \text{ without a collision.} \tag{21.15}$$

Obviously, each time there is a collision, the number $n_{\text{free}}$ decreases. In general, processes like this, where we can write a simple relation for the rate of change of a quantity, are well-modeled by differential equations. So, in the dilute limit,

$$n_{\text{free}}(\boldsymbol{x} + \Delta\boldsymbol{x}) - n_{\text{free}}(\boldsymbol{x}) = -\sigma \rho n_{\text{free}}(\boldsymbol{x}) \Delta\boldsymbol{x}, \tag{21.16}$$

which leads to the differential equation

$$\frac{\mathrm{d}n_{\text{free}}}{\mathrm{d}\boldsymbol{x}} = -\sigma \rho n_{\text{free}}(\boldsymbol{x}). \tag{21.17}$$

This is a linear ordinary differential equation[*], with solution

$$n_{\text{free}}(\boldsymbol{x}) = n_0 e^{-\sigma \rho x}. \tag{21.18}$$

That is, the number of particles without a collision decays exponentially as the particles move through space.

We can use the information contained in $n_{\text{free}}$ to compute the typical distances travelled without a collision, and, using the average particle velocity, the typical time per collision. The "mean free path" $\ell$ describes the typical distance without a collision, which is defined by

$$\ell = \frac{\langle v \rangle}{f_{\text{coll}}}, \tag{21.19}$$

the average velocity divided by the collision frequency. From what we have already derived, it is easy to see that

$$f_{\text{coll}} = \sqrt{\frac{8 k_{\text{B}} T}{\pi m}} \sigma \rho, \tag{21.20}$$

which, now accounting for the relative speed of two particles of the same mass (using the reduced mass $\mu = m/2$ in the expression for the average speed)

$$f_{\text{coll}} = \sqrt{2} \langle v \rangle \sigma \rho. \tag{21.21}$$

Hence, the mean free path is

$$\ell = \frac{1}{\sqrt{2} \sigma \rho} \tag{21.22}$$

---

[*] You should know the solution as an initial value problem immediately!

We can use this result to estimate the typical mean free path for a *near ideal* gas (noting that a truly ideal gas would have no collisions). Using the equation of state $\beta p = \rho$, we have

$$\ell_{\text{ideal}} = \frac{k_{\text{B}}T}{\sqrt{2}\sigma p}. \tag{21.23}$$

Because the typical time per collision is just

$$\tau_{\text{free}} = \frac{\text{mean free path}}{\text{average velocity}} = \frac{\ell}{v_{\text{avg}}}. \tag{21.24}$$

A back of the envelope calculation using $\sigma = 5$ Å and $\ell = 5 \times 10^{-8}$ m leads to $10^{10}$ collisions per second.

The rate at which particles collide determines the rate of reactions. We now consider two particles moving relative to each other—this two body problem can be reduced to a single body problem in which the reduced mass is the single variable that moves. Due to this, we work with the relative velocity, which on average is

$$v_{\text{rel}} = \sqrt{\left(\frac{8k_{\text{B}}T}{\pi\mu}\right)} \tag{21.25}$$

where $\mu$ is the reduced mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{21.26}$$

For a pair of particles, the collision probability is thus

$$\sigma v_{\text{rel}} \times [\frac{1}{2}\rho^2] \tag{21.27}$$

which gives us an explicit expression for the reaction rate (now using molar quantities),

$$k_D = N_{\text{A}}\sigma v_{\text{rel}}. \tag{21.28}$$

# Lecture 22 Diffusion

## Recap

1. We derived the average speed,
$$\langle v \rangle = \sqrt{\frac{8k_\mathrm{B}T}{\pi m}} \tag{22.1}$$
2. We discussed diffusion limited problems in the gas phase, where $k_D$ dominates the timescale of a reaction.
3. To compute $k_D$, we computed the collision density
$$\varrho_\text{collision} = \frac{1}{2}\sigma v_\text{rel}\rho^2, \tag{22.2}$$
which is the number of collisions per unit volume per unit time for identical particles. For particles of types $A$ and $B$, the analogous expression is
$$\varrho_\text{collision} = \sigma_{AB} v_\text{rel}\rho_A\rho_B. \tag{22.3}$$
Then, noting that $\rho = [X]/N_\mathrm{A}$, the rate is simply
$$k = N_\mathrm{A}\sigma v_\text{rel}. \tag{22.4}$$

## Goals for today

1. In the gas phase, motion is transiently ballistic. The situation in a liquid is quite different, but diffusion equations model this behavior well. Today we will focus on diffusion in liquid states.
2. Derive Fick's law.
3. Solve the diffusion equation.

## 22.1 A simplistic view of single-particle diffusion

We will consider a collection of particles in a condensed phase system, for example, protein molecules in a solvent like liquid water[*]. If we imagine preparing the system so that all the proteins are concentrated in a very small volume, we intuitively know that the material will *diffuse* apart. Let's consider

---

[*] protein or otherwise in what follows I will call everything a "particle"

a very simple model of this process in one dimension. Assume that we have a large number $N(x)$ particles at position $x$. These particles move with a typical velocity of magnitude $v$ each unit of time $\tau$. The typical displacement is $\pm\delta = \pm v\tau$ and we assume that each displacement is statistically independent—it is no more likely to move towards the left or the right at any given time, regardless of the past. How does the distribution of particles evolve on average?

To assess this, let us compute the flux of particles from $x$ to $x+\delta$, which is the number of particles that move through a unit area per unit time. We know that the number of particles moving to $x+\delta$ through a plane of area $A$ separating $x$ and $x+\delta$ is simply $\frac{1}{2}N(x)$. There is a flow in the opposite direction, as well; the number of particles moving towards $x$ is $\frac{1}{2}N(x+\delta)$. Thus, the flux is simply

$$J(x) = \frac{N(x) - N(x+\delta)}{2A\tau}. \tag{22.5}$$

It is more convenient to write this expression in terms of the density; to do so, we just write

$$\begin{aligned} J(x) &= \frac{\delta^2}{\delta^2} \times \frac{N(x) - N(x+\delta)}{2A\tau}, \\ &= \frac{\delta^2}{2\tau}\frac{1}{\delta}\left(\frac{N(x) - N(x+\delta)}{2A\tau}\right), \\ &= -\frac{\delta^2}{2\tau}\left(\frac{\rho(x+\delta) - \rho(x)}{\delta}\right). \end{aligned} \tag{22.6}$$

If we take the limit that $\delta \to 0$, then we arrive at a classical result known as Fick's law,

$$J(x) = -D\frac{\partial \rho}{\partial x}, \tag{22.7}$$

where we have defined $D = \delta^2/2\tau$.

If we think about the trajectory of a single particle in this system, at time $t = n\tau$ its mean-squared displacement satisfies the recursive relation

$$\langle x_i(n\tau)^2\rangle = \langle x_i((n-1)\tau \pm \delta)^2\rangle = \langle x_i((n-1)\tau)^2\rangle + \delta^2. \tag{22.8}$$

In other words, the mean-squared displacement grows at a rate $\delta^2/\tau$, which is exactly the definition we gave for the diffusion coefficient (the slope of the mean-squared displacement plot was $2Dt$).

## 22.2 Diffusion equation

Fick's law tells us that the flux is determined by a local concentration gradient. Of course, from this information we can deduce how the density changes as a function of time. Throughout, we will need to be mindful of the fact that the total density is constant. We will assume that

$$\int_{-\infty}^{\infty} \rho(x)dx = C. \tag{22.9}$$

Locally (i.e., at a particular position $x$) the density can change. In fact, we know precisely how it changes. The change in density at position $x$ is just the "number-out" minus the "number-in", which is given by

$$\frac{\rho(x,t+\tau) - \rho(x,t)}{\tau} = \frac{(J(x) - J(x+\delta))A\tau}{A\delta\tau}. \tag{22.10}$$

We again take the limit that $\delta$ is small, but we also consider $\tau \to 0$. Hence, we obtain

$$\begin{aligned} \lim_{\tau \to 0, \delta \to 0} \frac{\rho(x, t+\tau) - \rho(x, t)}{\tau} &= \frac{(J(x) - J(x+\delta))A\tau}{A\delta\tau} \\ \partial_t \rho &= -\frac{\partial \cdot J}{\partial x \cdot} \end{aligned} \tag{22.11}$$

Because we already have an equation for the flux $J$ in terms of $\rho$, we can write the partial differential equation above purely in terms of $\rho$.

The diffusion equation

$$\partial_t \rho(x, t) = D \frac{\partial^2 \rho}{\partial x^2}. \tag{22.12}$$

Solutions of this equation will tell us how a concentration profile will diffuse as a function of time. In three dimensions, the equation has essentially the same form. The flux is written component-wise

$$\vec{J} = (J_x, J_y, J_z) \tag{22.13}$$

and the Fick's law still holds component-wise, as well. This means that we can relate the mass flux to the density gradient,

$$\vec{J} = -D \nabla \rho. \tag{22.14}$$

Just as before, the total change in density at a point $x$ will just be the sum of the change in fluxes along each axis. That is,

$$\begin{aligned} \partial_t \rho(\boldsymbol{x}, t) &= -\left( \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right) \\ &= D \nabla \cdot \nabla \rho, \\ &\equiv D \Delta \rho. \end{aligned} \tag{22.15}$$

Given a function $f$, the Laplacian operator $\Delta$ is the divergence of the gradient of $f$.

$$\Delta f = \nabla \cdot \nabla f = \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \tag{22.16}$$

Solving the diffusion equation requires that we specify initial conditions and boundary conditions. Just as in the thought experiment conducted in the first section of these notes, let us assume that, initially,

$$\rho(x, 0) = \delta(x). \tag{22.17}$$

For simplicity, we will assume that the particles can diffuse freely along the $x$-axis (i.e., there is no boundary).

How do we solve this PDE? We can simplify the problem by incorporating some physically relevant information. The first piece of information that we consider is the fact that the solution is translationally invariant—this means, if we shift everything (including the initial condition) the same function will remain a solution. In other words, how the density profile evolves as a function of time does not depend on where it is. Mathematically, this means that

$$\rho(x,t) \text{ and } \rho(x - x_{\text{shift}}, t) \tag{22.18}$$

are both solutions to (23.2).

When we have translational invariance, it is useful to work in Fourier space. If we write an expression for the density in terms of its Fourier transform,

$$\rho(x,t) = \int \hat{\rho}_k(t) e^{ikx} dk \tag{22.19}$$

we see that we can write the partial differential equation in terms of the independent Fourier modes at frequency $k$. The equation that we need to solve is thus

$$\partial_t \hat{\rho}_k(t) = -Dk^2 \hat{\rho}_k(t) e^{ikx} \tag{22.20}$$

but because $\hat{\rho}_k$ depends only on $t$, we have an ordinary differential equation

$$\frac{\mathrm{d}\hat{\rho}_k(t)}{\mathrm{d}t} = -Dk^2 \hat{\rho}_k(t) \tag{22.21}$$

the solution as an initial value problem is simply

$$\hat{\rho}_k(t) = e^{-Dk^2 t} \hat{\rho}_k(0). \tag{22.22}$$

Computing $\hat{\rho}_k(0)$ is straightforward because

$$\begin{aligned} \hat{\rho}_k(0) &= \int_{-\infty}^{\infty} \rho(x,0) e^{-ikx} dx \\ &= \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx \\ &= 1. \end{aligned} \tag{22.23}$$

At this point, we have a solution to the diffusion equation (but it is still written in terms of the Fourier components $\hat{\rho}_k$). To get to a solution in real space, we need to compute the inverse Fourier transform:

$$\begin{aligned} \rho(x,t) &= \int_{-\infty}^{\infty} \hat{\rho}_k(t) e^{ikx} dk \\ &= \int_{-\infty}^{\infty} e^{-Dk^2 t + ikx} dk. \end{aligned} \tag{22.24}$$

This integral is Gaussian—the kind we know and love—but it is also an integral in the complex plane, which is going to add some subtlety. First, let's "complete the square", which makes identifying the mean and variance easy:

$$-Dk^2 t - ikx = -Dt(k - \frac{ix}{2Dt})^2 - x^2/4Dt. \tag{22.25}$$

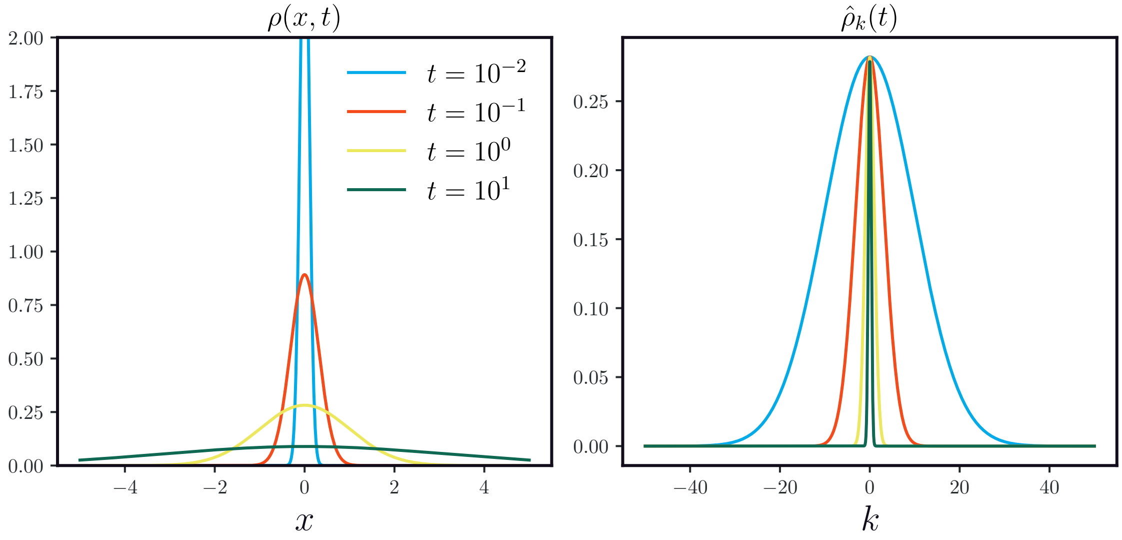


Figure 22.1: Solution of the diffusion equation in real space and in Fourier space.

Then,

$$\rho(x,t) = e^{-x^2/4Dt}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dt(k-ix/2Dt)^2}\,dk. \tag{22.26}$$

This remaining Gaussian integral requires the use of Cauchy's theorem of complex integration.* However, we do not really need to compute it: we now know what the shape of the distribution as a function of $x$ is at time $t$, and the density must be normalized. Thus,

$$\int_{-\infty}^{\infty} e^{-Dt(k-ix/2Dt)^2}\,dk = \sqrt{\frac{\pi}{Dt}}. \tag{22.27}$$

As a result, we get the normalized expression for $\rho$, which reads

$$\rho(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt}. \tag{22.28}$$

---

*Define a rectangular contour that intersects $ix/2Dt$ with the opposite edge along the real axis.

### Lecture 23 Diffusion limited rates

## Recap

1. We derived a classical result known as Fick's law,

$$J(x) = -D\frac{\partial \rho}{\partial x}, \tag{23.1}$$

2. We used the conservation of mass to derive

$$\partial_t \rho(x,t) = D\frac{\partial^2 \rho}{\partial x^2}. \tag{23.2}$$

3. We solved the diffusion equation in Fourier space for a 1D system with free boundaries,

$$\hat{\rho}_k(t) = e^{-Dk^2 t}\hat{\rho}_k(0) \tag{23.3}$$

which lead to the Gaussian real space solution

$$\rho(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-x_0)^2}{4Dt}}. \tag{23.4}$$

## Goals for today

1. Steady state solutions for common chemical and biological settings.
2. Discuss Reynold's numbers.
3. Discuss the Stoke's Einstein relation.

## 23.1 Diffusion equation at steady state

In one dimension, with the simplest possible initial condition and boundary conditions, it took some work to solve the diffusion equation. In general, it is challenging to compute time-dependent solutions for $\rho(\boldsymbol{x},t)$ when the geometry of the boundary conditions is complex. It is easier often much easier to assess the behavior at "steady-state". This nomenclature means that the density is no longer changing; that is, we must solve

$$\partial_t \rho(\boldsymbol{x},t) = 0 \qquad \Longrightarrow \qquad D\Delta\rho_{\mathrm{ss}}(\boldsymbol{x}) = 0. \tag{23.5}$$

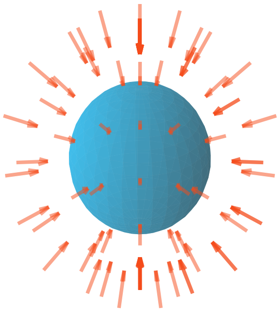


Figure 23.1: The geometry of the diffusion to capture problem.

In the second equation, the steady-state density no longer depends on time.

For the case we were able to solve last time, we had no boundary constraints. This means that

$$D\Delta\rho = 0 \qquad \Longrightarrow \qquad \rho(\boldsymbol{x}) = \rho_\infty, \tag{23.6}$$

which is just the constant bulk density.

Geometry plays a key a role in determining steady-state density profiles and there are many interesting examples that are highly relevant for biology and chemistry. Diffusion to capture is one such example. Consider an absorbing sphere of radius $\ell$, as depicted in Fig. 23.1. Particles are destroyed when they arrive at the surface of the sphere. Hence, the density at distance $\ell$ from the origin is

$$\rho(\boldsymbol{x}) = 0 \quad \|\boldsymbol{x}\| = \ell. \tag{23.7}$$

Infinitely far from the absorbing sphere, the density approaches the bulk density $\rho_{\text{bulk}}$.

Now, we must solve

$$D\Delta\rho(\boldsymbol{x}) = 0. \tag{23.8}$$

Fortunately, there's spherical symmetry in this problem, so the solution will be the same along a vector of magnitude $r$ away from the origin. That is, if we define

$$r^2 = x^2 + y^2 + z^2 \tag{23.9}$$

and then write $\rho$ as a one-dimensional function of $r$ we can write an equation for the flux,

$$J(r) = -D\frac{\partial\rho}{\partial r}. \tag{23.10}$$

Because we know that $\rho(r) \to \rho_{\text{bulk}}$ as $r \to \infty$, let's make a guess and verify[*] that it gives the right behavior:

$$\rho(r) \propto \left(1 - \frac{\ell}{r}\right) \rho_{\text{bulk}}. \tag{23.11}$$

Then, the flux is simply along one vector of magnitude $r$ is

$$J(r) = -D\rho_{\text{bulk}} \frac{\ell}{r^2}. \tag{23.12}$$

We conclude that the net inward flux is

$$-J(\|\boldsymbol{x}\| = \ell) = 4\pi\ell D\rho_{\text{bulk}}. \tag{23.13}$$

By dimensional analysis, we can check that this is indeed a flux (units: of mass per unit area per unit time).

One comment about this result: with a radial increase like this, the mass increases in proportion to $r^2$. However, the concentration gradient is a decreasing function of $r$. That is why the flux is a linear, not quadratic, function of $\ell$.

## 23.2 Estimating rates

Let's now return to the question of estimating rates. Suppose we have a process where large particles $B$ react with small particles $A$ when the two collide. First, we write

$$[A] = N_{\text{A}}\rho_A \quad [B] = N_{\text{B}}\rho_B \tag{23.14}$$

and define an interaction radius $\ell_{AB}$ to be the distance at which the two particles react. The reaction rate is just the number of $A$ particles times the inward flux:

$$k_{AB}[A][B] = N_A 4\pi D_{AB}\ell_{AB}[A][B]. \tag{23.15}$$

For a homogenous system, using Stoke's law $4\pi\eta\ell^{\star}$, we get,

$$k_{AB} = \frac{4N_A k_{\text{B}}T}{\eta} \tag{23.16}$$

which is the diffusion limited rate provided by what is commonly called Smoluchowski theory.

## 23.3 Diffusion with drift

If there is an force on the particles, they will move directionally. Let's try to account for this effect. Like last time, we consider displacements $\delta$ on a grid, however now we have

$$\delta = v_x\tau + \frac{1}{2m} f_x \tau^2 = (v_x + v_{\text{drift}})\tau. \tag{23.17}$$

---

[*] To do this, compute the Laplacian in spherical coordinates $\Delta f(r) = r^{-2}\frac{d}{dr}\left(r^2\frac{\mathrm{d}f}{\mathrm{d}r}\right)$.

[*] This holds when the particle is a similar size to surrounding particles.

We see that the drift velocity is

$$v_{\text{drift}} = \frac{f_x}{2m\tau^{-1}} = \frac{f_x}{\zeta}. \tag{23.18}$$

Here $\zeta$ is the coefficient of drag. The drift velocity is inversely related to the drag. We can compute the coefficient drag using the relation $\delta = v_x \tau$. Because $1/\tau = v_x/\delta$,

$$\begin{aligned}\zeta &= \frac{2mv_x}{\delta} \\ &= \frac{2mv_x^2}{\delta v_x} = \left(\frac{\delta^2}{2\tau}\right)^{-1} \times mv_x^2\end{aligned} \tag{23.19}$$

The average value of $\frac{1}{2}mv_x^2$ is $\frac{1}{2}k_{\text{B}}T$ by equipartition, so

$$D = \frac{k_{\text{B}}T}{\zeta} \tag{23.20}$$

This result was derived by Einstein in 1905 and is known as the Einstein relation.

Fluid mechanics provides an independent account of the drag forces on a particle. If we consider a particle moving through a viscous medium at constant velocity $v$, then Stokes' law states that the drag force is linearly proportional to the viscosity of the medium. More precisely, for a sphere of radius $\ell$,

$$f_{\text{drag}} = 6\pi\eta\ell v \tag{23.21}$$

where $\eta$ is the viscosity. This means that the diffusion coefficient is (according to "Stokes-Einstein") relation,

$$D_{\text{sphere}} = \frac{k_{\text{B}}T}{6\pi\eta\ell}. \tag{23.22}$$

As a result, if you can measure the diffusion coefficient (e.g., by microscopy) in a medium of known viscosity, then you can determine the size of a particle. Stoke's law can also be extended to more complex shapes.

The magnitude of viscous forces relative to inertial forces has an enormous consequence for dynamics. We can quantify this magnitude using a dimensionless quantity called the Reynold's number.

$$\text{Re} = \frac{v\ell\rho_{\text{G}}}{\eta}. \tag{23.23}$$

Approximating $\eta_{\text{H}_2\text{O}}$ as $10^{-2}$ g/(cm s) and $\rho_{\text{G}} = 1$ g/cm$^3$, we can do some back of the envelope calculations. For a small fish,

$$\text{Re} = \frac{(v = 10\text{cm/s}) \times (\ell = 10\text{cm})\rho_{\text{G}}}{\eta} \approx 10^4, \tag{23.24}$$

this means that inertia is much more important! However, for a bacterium,

$$\text{Re} = \frac{(v = 10^{-3}\text{cm/s}) \times (\ell = 10^{-4}\text{cm})\rho_{\text{G}}}{\eta} \approx 10^{-5}, \tag{23.25}$$

meaning that the viscous forces dominate. These are very different hydrodynamic environments. If you push a bacterium, it bounces right back.

# Lecture 24 Activated rates

## Recap

1. We looked at diffusion equations at steady-state,
$$\partial_t \rho(\boldsymbol{x}, t) = 0 \tag{24.1}$$
and solved one such equation with spherical symmetry.

2. Discussed the Einstein relation
$$D = k_{\mathrm{B}} T / \zeta \tag{24.2}$$
where $\zeta$ is the drag coefficient.

3. For a spherical solute of radius $\ell$ surrounded by a fluid with viscosity $\eta$, Stokes' Law says that the coefficient of drag is $\zeta = 6\pi\eta\ell$. This leads to the Stokes-Einstein relation,
$$D = \frac{k_{\mathrm{B}} T}{6\pi\eta\ell} \tag{24.3}$$
which relates the size (hydrodynamic radius) of the particle to its diffusion coefficient.

## Goals for today

1. Introduce transition state theory for activated processes.

2. Derive the Eyring equation.

## 24.1 Recap of Diffusion-limited vs Activated processes

Let us take the canonical example of $A + B \to AB$ as the reaction of interest. As we have previously discussed, if we write an equation for the rate of product formation by assuming steady-state of the transition state $AB^{\ddagger}$, we obtain a rate
$$\text{Rate} = k[A][B] = \frac{k_{\mathrm{a}} k_D}{k_{\mathrm{a}} + k_{\mathrm{D}}^r}[A][B]. \tag{24.4}$$

In the diffusion-limited ($k_D \ll k_{\mathrm{a}}$) regime, the overall rate was determined by the rate of encounter between reactants. We introduced several models of encounter

1. *Gas phase* In the gas phase, we modelled encounters by considering the cross-sectional area of colliding particles. This led to an expression,

   $$k_D = 4\sigma_{BC} N_{\mathrm{A}} \langle v_{\mathrm{rel}} \rangle \tag{24.5}$$

   where $\sigma_{BC} = \pi r_{BC}^2$ and $r_{BC} = r_B + r_C$ and $v_{\mathrm{rel}}$ is the relative speed.

2. *Liquid phase* In the liquid phase, we considered an absorber in a fluid of viscosity $\eta$ and found that the rate of encounter was

   $$k_D = \frac{4N_{\mathrm{A}} k_{\mathrm{B}} T}{\eta} \tag{24.6}$$

   We obtained this expression by deriving the diffusion equation and solving it at steady-state with appropriate boundary conditions.

In the limit where activation barriers determine the overall rate, we made a heuristic argument that the relative height of the barrier would play a role in determining the rate. This, of course, is because the probability of reaching the transition state, denoted by $AB^{\ddagger}$, is related to the height of the barrier. The probability of a state is not *a priori* related to the dynamics, though. We also need information about the flux through the transition state.

## 24.2 Transition state theory

Transition state theory is a surprisingly robust theory, but it does require a few assumptions.

- There is a well-defined "dividing surface" separating products and reactants.
- The evolution takes place on a single Born-Oppenheimer PES.
- Equilibrium between $A + B \rightleftharpoons AB^{\ddagger}$.
- No "re-crossing" $AB^{\ddagger} \to AB$ is final.

Though these assumptions seem quite strong, TST is often predictive outside the regime covered by these assumptions.

When is a rate constant a meaningful concept? The notion implies exponential kinetics.

$$\rho_A(t) = \rho_A(0) e^{-k_{\mathrm{TST}} t} \tag{24.7}$$

The rate $k_{\mathrm{TST}}$ sets the rate at which the population of $A$ molecules is decaying. First-order kinetics, requires separation of timescales. This occurs so long as the barrier is sufficiently large relative to $k_{\mathrm{B}}T$. As a rough guide, Baron Peters suggests $\approx 5k_{\mathrm{B}}T$ barriers are sufficient for the rate constant to be a meaningful concept.

Let's use the assumptions we have made thus far to build an expression for $k_{\mathrm{TST}}$

$$\frac{[AB^{\ddagger}]}{[A][B]} = K_{\mathrm{eq}}^{\ddagger} \equiv e^{-\beta \Delta G^{\ddagger}} \tag{24.8}$$

We can always compute the equilibrium constant as a ratio of partition functions, so

$$\frac{[AB^{\ddagger}]}{[A][B]} = \frac{Z_{AB^{\ddagger}}}{Z_A Z_B}. \tag{24.9}$$

To make progress, we make yet another assumption. There is a "reaction coordinate" $q$ that is the slowest relaxing degree of freedom, so it is the rate limiting variable. Near the transition state, we can factor out this contribution and write

$$K_{\text{eq}}^{\ddagger} = \frac{\tilde{Z}_{AB^{\ddagger}}}{\tilde{Z}_A \tilde{Z}_B} \tilde{Z}_{\text{TS}}(q) \tag{24.10}$$

Here, $\tilde{Z}_{\text{TS}}(q)$ is the partition for the reaction coordinate. The overall rate of product formation is determined by the flux through the transition state,

$$k[A][B] = \omega^{\ddagger}[AB^{\ddagger}] = \omega^{\ddagger} K_{\text{eq}}^{\ddagger}[A][B]. \tag{24.11}$$

Immediately, we obtain the Arrhenius-type scaling for the rate constant,

$$k \propto e^{-\beta \Delta G^{\ddagger}}. \tag{24.12}$$

But what is $\omega^{\ddagger}$ and how should we compute it?

At the transition state, a reasonable model of $\tilde{Z}_{\text{TS}}(q)$ is to consider it to be a one-dimensional particle in a box—it is an isolated degree of freedom on an essentially flat free energy landscape. With this model, the transition state partition function function becomes

$$\tilde{Z}_{\text{TS}}(q) = \frac{\delta q}{\lambda_{\text{T}}} \tag{24.13}$$

where

$$\lambda_{\text{T}} = \frac{h}{\sqrt{2\pi \mu_q k_{\text{B}} T}}, \tag{24.14}$$

and $\mu_q$ is the effective mass of the reaction of the reaction coordinate.

The overall rate that we wish to compute is determined by the speed of the reaction coordinate at the transition state and the concentration of the transition state,

$$k_{\text{TST}} = \omega^{\ddagger}[AB^{\ddagger}] \tag{24.15}$$

where $\omega^{\ddagger}$ quantifies the frequency of exits from the transition state. The microscopic expression for this frequency accounts for the speed of the reaction coordinate, the distance it must travel (the "width" of transition state), and the fact that only half the trajectories initiated from the top of the barrier will react, since there is an equal probability of positive and negative velocities. Putting these pieces together we obtain

$$\omega^{\ddagger} = \frac{1}{2} \frac{\langle |\dot{q}| \rangle}{\delta q}. \tag{24.16}$$

The average of the magnitude of $\dot{q}$ is an *average speed*. We know how to compute this!

We have done this integral in three-dimensional, where it is more complicated, but let us evaluate in one dimension. We need to compute

$$\frac{\int_{-\infty}^{\infty} |\dot{q}| e^{-\frac{1}{2}\beta \mu_q \dot{q}^2} d\dot{q}}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\beta \mu_q \dot{q}^2} d\dot{q}}, \tag{24.17}$$

splitting this about $\dot{q} = 0$, we obtain

$$\frac{2}{\sqrt{2\pi\beta\mu_q}}\int_0^\infty \dot{q}e^{-\beta\mu_q\dot{q}^2/2}d\dot{q} = \frac{2}{\sqrt{\frac{2\pi}{\beta\mu_q}}}\frac{1}{\beta\mu_q} = \sqrt{\frac{2k_\mathrm{B}T}{\pi\mu_q}}. \tag{24.18}$$

We are now ready to compute the overall rate.

Using everything we have computed thus far, we obtain

$$\begin{aligned} k_\mathrm{TST} &= \frac{1}{2}\frac{\langle|\dot{q}|\rangle}{\delta q}K_{eq}^\ddagger \\ &= \frac{1}{2}\frac{\langle|\dot{q}|\rangle}{\delta q}\tilde{Z}_\mathrm{TS}(q)\frac{\tilde{Z}_{AB^\ddagger}}{\tilde{Z}_A\tilde{Z}_B}, \\ &= \frac{1}{2}\frac{\langle|\dot{q}|\rangle}{\delta q}\frac{\delta q}{\lambda_\mathrm{T}}\frac{\tilde{Z}_{AB^\ddagger}}{\tilde{Z}_A\tilde{Z}_B} \\ &= \frac{1}{2}\sqrt{\frac{2k_\mathrm{B}T}{\pi\mu_q}}\frac{\sqrt{2\pi\mu_q k_\mathrm{B}T}}{h}\frac{\tilde{Z}_{AB^\ddagger}}{\tilde{Z}_A\tilde{Z}_B}, \end{aligned} \tag{24.19}$$

After a couple of beautiful cancellations, we are left with

$$k_\mathrm{TST} = \frac{k_\mathrm{B}T}{h}e^{-\beta\Delta G^\ddagger}. \tag{24.20}$$

One final nota bene: if we include a reference volume for the partition function, there is a volumetric factor that multiplies the rate,

$$k = \frac{k_\mathrm{B}T}{h}\frac{\tilde{Z}_{AB^\ddagger}/V}{\tilde{Z}_A/V\,\tilde{Z}_B/V} = \frac{k_\mathrm{B}T}{h}V^{v-1}e^{-\beta\Delta G^\ddagger}, \tag{24.21}$$

where $v$ is the order of the reaction.

# Lecture 25 Beyond transition state theory

## Recap

1. Discussed activated processes and assumptions of transition state theory.
2. Derived the Eyring equation
$$k_{\mathrm{TST}} = \frac{k_{\mathrm{B}}T}{h} e^{-\beta\Delta G^{\ddagger}}. \tag{25.1}$$
If we include a reference volume for the partition function, there is a volumetric factor that multiplies the rate,
$$k = \frac{k_{\mathrm{B}}T}{h} \frac{\tilde{Z}_{AB^{\ddagger}}/V}{\tilde{Z}_A/V \tilde{Z}_B/V} = \frac{k_{\mathrm{B}}T}{h} V^{v-1} e^{-\beta\Delta G^{\ddagger}}, \tag{25.2}$$
where $v$ is the order of the reaction.

## Goals for today

1. Examine the breakdown of transition state theory.
2. Relate correlation functions to the time-dependent distribution of states during a reaction.

## 25.1 Rate theory from Thermodynamics

$$\Delta G^{\ddagger} = \Delta H^{\dagger} - T\Delta S^{\dagger} \tag{25.3}$$

$$k = \frac{k_{\mathrm{B}}T}{h} V^{v-1} e^{-\Delta H^{\dagger} + T\Delta S^{\dagger}}, \tag{25.4}$$

$$\log\left(\frac{\beta h k}{V^{(v-1)}}\right) = -\beta\Delta H^{\dagger} + \Delta S^{\dagger}, \tag{25.5}$$

The temperature dependence of the rate tell us the enthalpy of activation and the intercept of the line is the entropy of activation $\Delta S^{\dagger}/k_{\mathrm{B}}$. These thermodynamic relations were observed and proposed based purely on empirical observation.

## 25.2 Breakdown of Transition State Theory

Where do our assumptions break? One setting is protein folding: because the stability of transition states can be a strong function of temperature, it is possible to observe Eyring plots that suggest a negative enthalpy of activation for folding at high temperatures. This typically means a breakdown of first order kinetics, often due to dynamical recrossing that arises from stable intermediates.

Let us investigate this failure in the context of a simple isomerization reaction

$$A \rightleftharpoons B \tag{25.6}$$

$$\frac{\mathrm{d}[A]}{\mathrm{d}t} = k_{BA}[B] - k_{AB}[A] \tag{25.7}$$

$$\frac{\mathrm{d}[B]}{\mathrm{d}t} = -\frac{\mathrm{d}[A]}{\mathrm{d}t} \tag{25.8}$$

Because $[A]$ evolves as a function of time, it is not always equal to its equilibrium value, $[A]_{\mathrm{eq}}$. Let us call the deviation from equilibrium

$$\Delta c_A(t) = [A](t) - [A]_{\mathrm{eq}}. \tag{25.9}$$

$$\Delta c_A(t) = \Delta c_A(0) e^{-t/\tau} \tag{25.10}$$

where

$$\tau = \frac{1}{k_{AB} + k_{BA}} \tag{25.11}$$

implication:

$$\frac{\Delta c_A(t)}{\Delta c_A(0)} \to 1 \tag{25.12}$$

exponentially fast.

To observe this dynamics, we need a *nonequilibrium* initial condition, $[A](t) \neq [A]_{\mathrm{eq}}$.

Define an indicator function to keep track of the fraction of $A$ molecules. Using our reaction coordinate $q(\boldsymbol{x})$,

$$h_A(\boldsymbol{x}) \begin{cases} 1 \text{ if } q(\boldsymbol{x}) \in A \\ 0 \text{ otherwise} \end{cases}. \tag{25.13}$$

The total number of $A$ molecules is simply

$$n_A(\boldsymbol{x}_t^N) = \sum_{i=1}^{N} h_A(\boldsymbol{x}_i). \tag{25.14}$$

At equilibrium, the expected distribution is Boltzmann

$$\rho_{\mathrm{eq}}(\boldsymbol{x}) = Z^{-1} e^{-\beta \mathcal{H}(\boldsymbol{x})} \tag{25.15}$$

We prepare an initial condition by energetically favoring $A$,

$$\rho_0(\boldsymbol{x}^N) = Z^{-1}(\epsilon) e^{-\beta\mathcal{H}(\boldsymbol{x}^N)+\beta\epsilon n_A(\boldsymbol{x}^N)} \tag{25.16}$$

Imagine at $t=0$, we set $\epsilon=0$ and let the isomerization proceed. Then, the distribution evolves. Keeping track of the collection of molecules $\boldsymbol{x}_t^N$, we get a nonequilibrium average

$$\bar{n}_A(t) = Z^{-1}(\epsilon)\int n_A(\boldsymbol{x}_t^N) e^{-\beta\mathcal{H}(\boldsymbol{x}_0^N)+\beta\epsilon n_A(\boldsymbol{x}_0^N)} d\boldsymbol{x}_0^N \tag{25.17}$$

The partition function $Z(\epsilon)$ is

$$\int e^{\beta\epsilon n_A(\boldsymbol{x}_0^N)}\rho_{\text{eq}}(\boldsymbol{x}_0^N) Z_{\text{eq}} d\boldsymbol{x}_0^N = \left\langle e^{\beta\epsilon n_A}\right\rangle Z_{\text{eq}}. \tag{25.18}$$

Replacing $e^{-\beta\mathcal{H}}$ with $Z_{\text{eq}}\rho_{\text{eq}}$

$$\begin{aligned}\bar{n}_A(t) &= \frac{1}{\left\langle e^{\beta\epsilon n_A}\right\rangle Z_{\text{eq}}}\int n_A(\boldsymbol{x}_t^N) e^{\beta\epsilon n_A(\boldsymbol{x}_0^N)}\rho_{\text{eq}}(\boldsymbol{x}_0^N) Z_{\text{eq}} d\boldsymbol{x}_0^N \\ &= \frac{\left\langle n_A(t) e^{\beta\epsilon n_A(0)}\right\rangle}{\left\langle e^{\beta\epsilon n_A(0)}\right\rangle}\end{aligned} \tag{25.19}$$

While we are extracting information about nonequilibrium distributions, these are both equilibrium averages.

In fact, we can think about this in terms of time correlations. Define

$$\delta n_A(t) = n_A(t) - \langle n_A\rangle \tag{25.20}$$

Obviously, the equilibrium average of $\delta n_A$ is zero. However, the time-correlation function

$$\langle \delta n_A(0)\delta n_A(t)\rangle \equiv \mathcal{C}(t) \tag{25.21}$$

is non-zero. Onsager regression hypothesis: the way that a system approaches equilibrium is the same as the way it returns to equilibrium after a natural fluctuation. This holds "close to equilibrium", the linear response regime. Implication: This equilibrium time-correlation function is deeply related to the nonequilibrium dynamics

$$\frac{\delta\bar{n}_A(t)}{\delta\bar{n}_A(0)} = \frac{\mathcal{C}(t)}{\mathcal{C}(0)}. \tag{25.22}$$